AG Exercises

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These are a set of exercises I wrote while I was Super TA for Algebraic Geometry at the University of Arizona to complement Andreas Gathmann's course notes (available online). I claim no originality on any of the exercises. Some have references to where they come from, but I did not do a complete job specifying their source. Some are general knowledge, some are theorems from other books...There are also some that are questions I had that I never found a satisfactory answer to.

1 First Set

1. (a) Show that a closed subset of an irreducible space may be reducible.
   (b) Show that any non-empty open subset of an irreducible space is irreducible and dense.
   (c) Give an example of an open subset in a reducible space which is not dense.
2. Show that a space X is irreducible if and only if any two non-empty open sets intersect.
3. A Noetherian Hausdorff space must be a finite set of points. Source: Hartshorne ex.I.1.5.
4. A Noetherian space is *quasi*-compact, i.e., any open cover has a finite subcover. (It is not compact because the definition of compactness includes the condition that the space be Hausdorff, and the exercise above shows this rarely happens!) Source: Hartshorne ex.I.1.5
5. (a) Let k be algebraically closed. Find an $f \in k[x_1, \ldots, x_n]$ such that $Z(f)$ is irreducible even though f is reducible.
   (b) If k is algebraically closed and $f \in k[x_1, \ldots, x_n]$ is a square-free polynomial, show that $Z(f)$ is irreducible if and only if f is irreducible.
   (c) Give and example of polynomial $f \in \mathbb{R}[x, y]$ for which $Z(f)$ is reducible even though f is irreducible. Source: Hartshorne ex.I.1.12

6. Let $Y = Z(y^2 - xz, yz - y) \subseteq \mathbb{A}^3$. Show that Y is the union of three irreducible components. Describe them and find their prime ideals. Source: Hartshorne ex.I.1.3.

7. Decompose into irreducible components the closed set $Z(y^2 - xz, z^2 - y^3)$. Source: Shafarevich p. 40.

8. The curve $X = \{(t, t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^3$ is called the *twisted cubic.*

   (a) Find generators for the ideal of X.

   (b) Show that X is one dimensional.

9. Show that if k is not algebraically closed, then any closed subset X of $\mathbb{A}^n$ is a hypersurface (i.e., $X = Z(f)$ for some f). Source: Doug Ulmer.
   Hint: Show that for any m there is a $g_m \in k[t_1, \ldots, t_m]$ such that
   $$g_m(a_1, \ldots, a_m) = 0 \text{ iff } (a_1, \ldots, a_m) = \vec{0}.$$
   Start with $m = 2$.

10. (Finite sets are complete intersections) Let $X \subseteq \mathbb{A}^n$ be a finite set. Show that there are n polynomials such that $X = Z(f_1, \ldots, f_n)$. Source: Doug Ulmer.

11. Show that any irreducible curve in $\mathbb{A}^2$ of degree 2 can be parametrized by rational functions. This parametrization may be undefined at some points, and may miss finitely many points (for example, the Weierstrass substitution from calculus gives a parametrization of the circle).
    Hint: Think about the lines through a point.
    Hint: This won't work in general for higher degree curves.

12. Show that if k is algebraically closed then any homogeneous polynomial in two variables factors into linear factors.

13. Let $k = \mathbb{C}$ and $C = Z(f)$ with f irreducible be a curve in $\mathbb{A}^2$. Let $p \in C$ be a point, and consider the family lines through p given parametrically as
    $$\vec{r}(t) = p + t\vec{v},$$
    where $\vec{v}$ is a unit vector.

    (a) Show that $f(\vec{r}(t))$ vanishes with the same order at $t = 0$ except for finitely many $\vec{v}$ at which it vanishes with a higher order. We call this minimun order the multiplicity of p in C, and we call the union of the degenerate lines the tangent cone of C at p. For example:
       - $(0,0)$ is a point of multiplicity 1 on $(x-1)^2 + y^2 = 1$.
       - $(0,0)$ is a point of multiplicity 2 on $y^2 = x^3$ and the tangent cone at C is the line $y = 0$.

(b) Show that p is a point of multiplicity 1 if and only if $f_x(p)$ and $f_y(p)$ don't both vanish at p. We call there points the smooth points of p. Note that $(0,0)$ is not a smooth point of the curve $y^3 = x^5$ even though there is no apparent singularity in the graph of the curve over the real numbers.

(c) Show that at smooth points the tangent cone is a single line.

(d) Let $p = (0,0)$ and f_m be homogeneous part of f of smallest degree. Show that the tangent cone of C at p is the curve $f_m = 0$ (a union of lines by an exercise above).

14. Show that any reducible curve in $\mathbb{A}^2$ is singular.

2 Second Set

1. Show that $C : xy = 1 \subset \mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$.

2. At what points on the circle $x^2 + y^2 = 1 \subset \mathbb{A}^2$ is the rational function $(1-y)/x$ regular? Source: Shafarevich.

3. At what points of the curve $C : y^2 = x^3 + x^2 \subset \mathbb{A}^2$ is the rational function $t = y/x$ regular? Show that $y/x \notin A(C)$. Source: Shafarevich.

4. Prove that any map of the form $(x, y) \mapsto (ax, by + p(x))$ with $a, b \neq 0$ and $p(x) \in k[x]$ is an automorphism of $\mathbb{A}^2$. Source: Shafarevich.

5. Which of the following are isomorphic over $\mathbb{C}$? Source: Gathmann

   (a) $\mathbb{A}^1$
   (b) $xy = 0$ in $\mathbb{A}^2$.
   (c) $x^2 + y^2 = 0$ in $\mathbb{A}^2$.
   (d) $y^2 = x^3 + x^2$ in $\mathbb{A}^2$.
   (e) $x^2 = y^3$ in $\mathbb{A}^2$.
   (f) $Z(y - x^2, z - x^3)$ in $\mathbb{A}^3$.

6. Give as much information as you can about the stalks of the structure sheaf at the origin in each of the examples above for which it makes sense (i.e. for the ones that are irreducible!). Find a minimum number of generators for the maximal ideal of each.

7. Show that any conic in $\mathbb{A}^2$ is isomorphic to either $xy = 1$ or $y = x^2$ over $\mathbb{C}$ (excluding the degenerate cases).

8. Go through all the details of example 2.2.3 in the text.

9. Let $X \subset \mathbb{A}^n$ be an affine variety, and $Y \subset X \subset \mathbb{A}^n$ with Y closed and irreducible in X. Show that if $i : Y \to X$ is the inclusion, then $i^* : A(X) \to A(Y)$ is surjective. Therefore, regular functions on Y always come from regular functions from X. Compare this to when Y is open in X (for example $Y = X_f$ for some $f \in A(X)$).

10. Let U be a domain of $\mathbb{C}$ and define

$$F(U) = \{\text{Complex Analytic functions } f \text{on } U \mid zf'(z) = 1\}.$$

   (a) Show that F is a sheaf on $\mathbb{C}$ with restriction of functions.

   (b) Show that the stalk of F at $z = 0$ is empty, and any other stalk is non-canonically isomorphic to $\mathbb{C}$. Source: Kempf.

11. Let F be a presheaf on a topological space X, U and open set of X, and let $\sigma, \tau \in F(U)$.

   (a) Show that $\sigma_p = \tau_p$ in every stalk F_p for $p \in U$ if and only if there is an open cover $U = \cup U_\alpha$ such that $\sigma\mid_{U_\alpha} = \tau\mid_{U_\alpha}$.

   (b) Show by an example that $\sigma_p = \tau_p$ in every stalk F_p for $p \in U$ does not imply that $\sigma = \tau$ on $F(U)$. Source: Kempf

12. Show that the zariski topology of $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ is not the product topology of the zariski topologies of the $\mathbb{A}^1$'s.

13. Show that $\mathbb{A}^2 - \{(0,0)\}$ is not an affine variety.

3 Third Set

1. Let $f : X \to Y$ be a morphism of varieties. Show that the preimage of any Y_g is an X_h. Shource: Shafarevich.

2. Let X be an affine variety. Show that $X_f \cap X_g$ is affine for every $f, g \in \mathcal{O}_X(X)$.

3. Let $\phi : X \to Y$ be a morphism of varieties.

   (a) Show that ϕ induces a map in the stalks $\phi_p^\star : \mathcal{O}_{Y,\phi(p)} \to \mathcal{O}_{X,p}$ for all $p \in X$.

   (b) Show that ϕ is an isomorphism if and only if the following two conditions hold:

      - ϕ is a homeomorphism.
      - $\phi_p^\star$ is an isomorphism for every $p \in X$.

   (c) Show that if the image of X is dense in Y then $\phi_p^\star$ is injective for all $p \in X$. Source: Hartshorne.

4. Let X be a variety and U, V open subsets which are affine. Show that $U \cap V$ is also affine. Source: Shafarevich.

   Hint: Show that the map $U \cap V \to X \times X : u \to (u, u)$ is a morphism and use the fact that X is separated. (i.e. a prevariety which is a variety).

5. Show that the image $f(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$ of the Segre embedding is not contained in any linear subspace of $\mathbb{P}^N$. Source: Shafarevich.

6. Show that any two curves in $\mathbb{P}^2$ intersect. Source: Hartshorne.

7. Prove that any morphism $\mathbb{P}^1 \to \mathbb{A}^n$ is constant.

8. Show that an affine variety is complete if and only if it is one point.

9. Show that if $k = \mathbb{C}$ and $X \subset \mathbb{P}^n$ is closed, then $\mathbb{P}^n - X$ is path conected. Source: Doug Ulmer. Hint: Start with $n = 1$.

10. Let X be a variety and $p \in X$. Show that there is a one to one correspondence between the prime ideals of $\mathcal{O}_{X,p}$ and the closed subvarieties of X that contain p. Source: Hartshorne.

4 Fourth Set (on projections and points in general position)

Definition 1 An automorphism $\mathbb{P}^n \to \mathbb{P}^n$ which is induced by a linear automorphism of $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ is called a *projective equivalence.*

Two closed sets of $\mathbb{P}^n$ are said to be *projectively equivalent* if there is projective equivalence of $\mathbb{P}^n$ taking one to the other.

Definition 2 A collection of points $\{p_i\} \subset \mathbb{P}^n$ is said to be *independent* if the corresponding vectors in $\mathbb{A}^{n+1}$ are linearly independent, and it is said to be *dependent* otherwise.

A collection of points $\{p_i\} \subset \mathbb{P}^n$ is said to be in *general position* if no subcollection of $n + 1$ of them is dependent.

1. Show that any two linear subspaces of $\mathbb{P}^n$ of the same dimension are projectively equivalent.

2. Show that any two irreducible conics in $\mathbb{P}^2$ are projectively equivalent.

3. Show that any collection of $n+2$ points in general position in $\mathbb{P}^n$ is projectively equivalent to the collection

$$\begin{aligned} p_1 &= [1:0:\ldots:0] \\ p_2 &= [0:1:\ldots:0] \\ &\vdots \\ p_{n+1} &= [0:0:\ldots:1] \\ p_{n+2} &= [1:1:\ldots:1] \end{aligned}$$

4. Show that two collections of $4 = 1+3$ points in $\mathbb{P}^1$ in general position are projectively equivalent if and only if they have the same cross-ratio. Source: Harris. Hint: Show that the cross ratio of p_1, p_2, p_3, p_4 is the image of p_4 under the projective equivalence of $\mathbb{P}^1$ taking p_1, p_2, p_3 to $1, \infty, 0$ respectively.

5. Let p, q be distinct points in $\mathbb{P}^n$. Show that the "parametric equation" for the line $\overline{pq}$ through p and q is given by the map

$$\begin{aligned} \mathbb{P}^1 &\rightarrow \mathbb{P}^n \\ [s:t] &\mapsto sp + tq \end{aligned}$$

6. Let H be an $(n-1)$-dimensional linear subspace H of $\mathbb{P}^n$ (so $H \cong \mathbb{P}^{n-1}$), and let $p \in \mathbb{P}^n$ a point which is not contained in H. The projection of $\mathbb{P}^n$ from p to H is the map

$$\begin{aligned} \pi_p : \mathbb{P}^n - \{p\} &\rightarrow H \\ q &\mapsto \overline{pq} \cap H \end{aligned}$$

where $\overline{pq}$ is the line through p and q. Show that up to a projective equivalence, the projection is given in coordinates by

$$[x_0 : x_1 : \ldots : x_n] \mapsto [x_0 : \ldots : x_{n-1} : 0].$$

7. Let C be a smooth conic in $\mathbb{P}^2$. Show that the projection of $\mathbb{P}^2$ from a point $p \in C$ when restricted to C extends to an isomorphism $C \cong \mathbb{P}^1$.

8. The twisted cubic is the closure in $\mathbb{P}^3$ of the parametric curve $\mathbb{A}^1 \rightarrow \mathbb{A}^3 : x \rightarrow (x, x^2, x^3)$.

   (a) Show the twisted cubic is the image the projective map

$$\begin{aligned} \mathbb{P}^1 &\rightarrow \mathbb{P}^3 \\ [s:t] &\mapsto [s^3 : s^2t : st^2 : t^3] \end{aligned}$$

(b) Show that the image of any map

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \mathbb{P}^3 \\ [s:t] & \mapsto & [p_0(s,t) : p_1(s,t) : p_2(s,t) : p_3(s,t)] \end{array}$$

with the p_i homogeneous polynomials of degree 3 is projectively equivalent to the twisted cubic as long as the p_i form a basis for the vector space of homogeneous polynomials of degree 3.

(c) We use the term *twisted cubic* for any of the curves above. Show that any finite set of points on a twisted cubic is in general position.

(d) Show that there is a unique twisted cubic in $\mathbb{P}^3$ through 6 points in general position. Source: Harris.

5 Fifth Set

1. Let $X \subset \mathbb{A}^n$ be an affine variety and $\overline{X} \subset \mathbb{P}^n$ be its projective closure. Show that $\overline{X}$ does not contain the hyperplane at infinity.

2. Show that a line in $\mathbb{P}^n$ is either contained in the hypersurface of degree d, or intersects it in exactly d points counting multiplicities.

3. Consider the following rational map from $\mathbb{P}^2$ to itself:

$$\begin{array}{rcl} \phi : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ [x:y:z] & \mapsto & [yz : xz : xy] \end{array}$$

   (a) Find all the points where ϕ is not defined and where ϕ is not injective.

   (b) Show that ϕ is its own inverse on an open subset of $\mathbb{P}^2$. Find the maximal open subset where this happens.

4. Show that in the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ the sets $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ get mapped to lines.

5. Let X be a closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ and let π_1 and π_2 be the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto its factors. Show that $\pi_1(X) = \pi_2(X) = \mathbb{P}^1$ unless X is a point or one of the lines of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. Source: Shafarevich.

6. Show that the homogeneous coordinate ring is not invariant under isomorphism by showing that even though $X = \mathbb{P}^1$ is isomorphic to a conic Y in $\mathbb{P}^2$ by the Veronese embedding, the homogeneous coordinate rings of X and Y are not isomorphic. Source: Hartshorne.

7. Show that any degree 2 hypersurface in $\mathbb{P}^n$ is projectively equivalent to the hypersurface defined by $x_0^2 + x_1^2 + \ldots + x_k^2 = 0$ for some unique $0 \leq k \leq n$.

8. Let $\mathbb{P}^N$ be the projectivization of the vector space of homogeneous degree d polynomials in $x_0, \ldots, x_n$. Show that the reducible degree d polynomials are a closed set of this $\mathbb{P}^N$.

9. (a) Show that one can map any set of three non-intersecting lines to any other set of non-intersecting lines in $\mathbb{P}^3$ using a linear transformation.

   (b) Show that given three non-intersecting lines L_1, L_2, L_3 in $\mathbb{P}^3$, the union of the lines in $\mathbb{P}^3$ intersecting the three lines is a smoth quadric surface in $\mathbb{P}^3$. Hint: Think of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ and three line on one ruling. Use Bezout.

   (c) Show that there are exactly two lines through any point p on a smooth quadric S in $\mathbb{P}^3$. Hint: These are the two lines of the ruling of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. Think of the tangent plane at the point.

   (d) Prove that there are exacly two lines (counting multiplicities) intersecting a fixed set of 4 non-intersecting lines in $\mathbb{P}^3$ that do not all lie on a quadric surface.

6 Sixth Set

1. Let R be a commutative ring, and $\mathfrak{m}$ be a maximal ideal.

   (a) Show that $R_{\mathfrak{m}} \cong R$ if R is a local ring.

   (b) Show by an example that $R_{\mathfrak{m}} \ncong R$ in general.

2. Let $R = k[x]/\langle x^2 \rangle$.

   (a) Show that $\mathrm{Spec}(R)$ only has one point.

   (b) Show that $R_{\langle x \rangle} \cong R$.

   (c) Is $\mathrm{Spec}(R)$ irreducible?

3. Let $X = \mathrm{Spec}(R)$ be an affine scheme, let $p \in X$ be a point and m_p be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. Show that $\mathcal{O}_{X,p}/m_p$ is isomorphic to the residue field $\kappa(p)$ at p defined by $\mathrm{Frac}(R/p)$.

4. Let R be an arbitrary communtative ring and $X = \mathrm{Spec}(R)$.

   (a) Show that X is quasi-compact (any cover has a finite subcover).

   (b) Find an example showing that X might not be Noetherian.

   (c) Show that X being noetherian is not directly related to R being noetherian by finding and example with X noetherian and R not noetherian. (Hartshorne II.2.13).

5. What are the irreducible components of Spec $k[x]/(x^2(x-1)^3)$?

7 Seventh Set

1. Let F be an arbitrary field and X a scheme. Show that giving a morphism $\text{Spec}(F) \to X$ is exactly the same as giving a point $p \in X$ and an inclusion of fields $\kappa(p) \hookrightarrow F$. (Hartshorne II.2.7)

2. (a) Show that there is a morphism ϕ from any scheme X to $\text{Spec}(\mathbb{Z})$.

   (b) Show that this morphism sends $p \in X$ to the characteristic of the residue field $\kappa(p)$ at the point.

3. **Definition** Let X be an arbitrary scheme and $p \in X$ a point. We define the tangent space to X at p to be the dual space of the $\kappa(p)$-vector space m_p/m_p^2 where m_p is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

   (a) Give the details of how m_p/m_p^2 gets the structure of a $\kappa(p)$-vector space.

   (b) Show that if $X = \text{Spec } R$ is affine and p is closed and corresponds to the maximal ideal M in R, then $M/M^2 \cong m_p/m_p^2$ as $\kappa(p) = R/M$ vector spaces.

   (c) Show that $X = Z(xy, xz, yz) \subseteq \mathbb{A}^3_{\mathbb{C}}$ and $Y = Z(xy(x-y)) \subseteq \mathbb{A}^2_{\mathbb{C}}$ are not isomorphic by computing their tangent spaces at closed points.

   (d) Show by an example that the conclusion in part (b) does not hold for non-closed points.

   (e) Let X be a scheme over a field k. Show that giving a morphism of k-schemes
   $$\phi : \text{Spec}\left(k[\epsilon]/\langle \epsilon^2 \rangle\right) \to X$$
   is exactly the same as giving a point $p \in X$ with $\kappa(p) = k$ and an element of T_p. (Hartshorne II.2.8)

8 Eighth Set

1. Let $\text{Spec } A \xrightarrow{f} \text{Spec } B$ be a morphism of affine schemes. Let Z be a closed subscheme of Spec B corresponing to an ideal I of B. Show that the scheme theoretic inverse of Z is defined by $(f^\star I)A$, the ideal generated by $f^\star I$ inside A.

2. Let $f : X \to Y$ be a morphism of schemes. Show that if $y \in Y$ is not in the image of f then $X \times_Y \text{Spec } \kappa(y)$ is empty by reducing to the affine case and showing that the tensor product of the rings is the zero ring

3. Let $f : \text{Spec } \mathbb{Z}[x] \to \text{Spec } \mathbb{Z}$ be the morphism induced from the inclusion. Describe all the fibers of f.

9 Ninth Set

1. Show that the homogeneous coordinate ring is not invariant under isomorphism by showing that even though $X = \mathbb{P}^1$ is isomorphic to a conic Y in $\mathbb{P}^2$ by the Veronese embedding, the homogeneous coordinate rings of X and Y are not isomorphic. Source: Hartshorne Ch I ex. 3.9.

2. Let C be a smooth conic in $\mathbb{P}^2_{\mathbb{R}}$ and p be a point on this conic. Show how to construct the tangent line to C at p with a straightedge only. Hint: Generalize Pascal's theorem by allowing two points to coincide.

3. Let C be an irreducible curve of degree d in $\mathbb{P}^n$. Show that C is contained in a linear subspace of $\mathbb{P}^n$ of dimension d (this is trivial if $d > n$).

4. Let C_1, C_2 be curves in $\mathbb{P}^2$ and p be a point in their intersection. Show that the length of the component of $C_1 \cap C_2$ at p is equal to $\dim_k \mathcal{O}_{\mathbb{P}^2,p}/\langle F_1, F_2\rangle$ where F_1 and F_2 are the polynomials defining C_1 and C_2 respectively.

5. (a) Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ be the degree d Veronese embedding, and let $p_1, \ldots, p_m$ be any set of distinct points in $\mathbb{P}^n$. Consider the matrix

$$A_d = \begin{bmatrix} --\nu_d(p_1)-- \\ --\nu_d(p_2)-- \\ \vdots \\ --\nu_d(p_{d+1})-- \end{bmatrix}$$

containing as rows (representatives in k^{N+1} of) the images of the points under the Veronese map. Show that if X is the collection of these points, then $h_X(d) = \text{rank } A_d$.

   (b) (From Harris (13.1)) Show that if X is a collection of m distinct points in $\mathbb{P}^n$ and $d \geq m-1$, then you have $h_X(d) = m$.

   (c) Prove that the image of any $d+1$ points under $v_d : \mathbb{P}^n \to \mathbb{P}^N$ is in general position.

6. (a) Let C_1, C_2 be distinct smooth cubic curves in $\mathbb{P}^2$. Assume that $C_1 \cap C_2$ consists of 9 distinct points $p_1, \ldots, p_9$. Show that any cubic passing through $p_1, \ldots, p_8$ also passes through p_9.

   (b) Show that the same conclusion holds of we allow some p_i to be equal as long as we count the intersection multiplicities.

   (c) Show that the conclusion of (b) holds even if we allow C_2 to be singular.

7. Let $C = Z(f)$ be a smooth cubic in $\mathbb{P}^2$, let Hf be the hessian of f. Let p be a point in $\mathbb{P}^2$ and let $X_p = Z(g)$ where $g = \nabla f(x) \cdot p$ where where $x = (x_0, x_1, x_2)$ are the coordinates in $\mathbb{P}^2$, and where we are taking for p a representative vector.

(a) Explain why $q \in C \cap X_p$ iff the tangent line to C at q goes through p.

(b) Show that $g = x^t Hf(p)x$, and conclude that X_p is a union of lines iff $p \in C$ is an inflection point.

(c) Show that if p is an inflection point, and $q \in C \cap X_p$ is different from p, then q is not an inflection point of C.

(d) Show that $Hf(q) \cdot q = 3\nabla f(q)$ for any point q, and conclude that if X_p is a union of lines, then one of these lines is the tangent to C at p.

(e) Show that if p is an inflection point, and $q \in C \cap X_p$ is different from p, then q is smooth in X_p and $T_{C,q} \neq T_{X_p,q}$. Use this to conclude that there are exactly 3 distinct points of C other than p whose tangent line contains p.

8. Let C be a smooth cubic defined by $f = 0$ in $\mathbb{P}^2$. Let p be a fixed point of C and use the notation $x = [x_0 : x_1 : x_2]$, $\hat{x} = (x_0, x_1, x_2)$ for projective coordinates, and a point associated to it. Show that the map sending the point x to the 3rd intersection point of the line through x and p is given by

$$\begin{array}{rcl} \phi : C - \{p\} & \to & C \\ x & \mapsto & [(\nabla f(x) \cdot \hat{p})\, \hat{p} - (\nabla f(p) \cdot \hat{x})\, \hat{x}] \end{array}$$

10 Tenth Set

1. Compute the geometric multiplicities of the irreducible components of Spec $k[x]/\langle x^2(x-1)^3 \rangle$ using the definition.

2. Do the same for Spec $k[x,y]/\langle x - 2y^2, (x-1)^1 + y^2 - 1 \rangle$.

11 Eleventh Set

1. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that ϕ is surjective if and only if for every open U in X and for every $\sigma \in \mathcal{G}(U)$ there is an open cover $U = \cup U_\alpha$ and elements $\tau_\alpha \in \mathcal{F}(U_\alpha)$ such that $\phi(U_\alpha)(\tau_\alpha) = \sigma\,|_{U_\alpha}$.

2. Show that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves on X, then
$$0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X)$$
is always exact.

3. (a) Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of quasi-coherent sheaves on a scheme X. Show that if $\mathcal{G}$ is locally free, then
$$0 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$
is exact.

(b) Does the above hold if $\mathcal{G}$ is only quasi-coherent?

4. Show that the Hom sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ defined by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \text{set of sheaf homomorphisms } \mathcal{F}\mid_U \to \mathcal{G}\mid_U$$

is a sheaf, and give it the structure of an $\mathcal{O}_X$-module.

5. If $\mathcal{F}_1 \cong \mathcal{F}_2$, then $\mathcal{H}om(\mathcal{F}_1, \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}_2, \mathcal{G})$

6. (a) Show that if $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, then

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{G}_p.$$

(b) If $\mathcal{F}$ and $\mathcal{G}$ are locally free, are the stalks of the Hom sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ are given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})_p = \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{F}_p, \mathcal{G}_p)?$$

7. Is the following true? If $\mathcal{F}$ is a locally free sheaf of rank r, then $\mathcal{F} \otimes_{O_X} k_p = k_p^{\oplus r}$.

12 Twelfth Set

1. Explicitly construct the vector bundle on $\mathbb{P}^n$ corresponding to $\mathcal{O}_{\mathbb{P}^n}(a)$ by giving a trivializing cover and finding the transition functions.

2. Supply all the details of the proof that $\mathcal{O}(a) \otimes \mathcal{O}(b) = \mathcal{O}(a+b)$ on $X = \mathbb{P}^n$ (example (7.1.19) in Gathmann is missing the fact that the map he defines is in fact an isomorphism, and is ignoring the fact that the tensor sheaf needs sheafification).

3. Show that $\bigwedge^m(\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_m)) = \mathcal{O}(a_1 + \ldots + a_m)$ on $X = \mathbb{P}^n$.

4. The Picard Group operation on line bundles:

   (a) Let $\mathcal{L}$ be a line bundle. Show that $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}$, where $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O})$.

   (b) Show that $\mathcal{O}(D)^\vee \cong \mathcal{O}(-D)$.

   (c) Show that $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \cong \mathcal{O}(D_1 + D_2)$

   (d) Use all the previous parts to explain why the set of isomorphism classes of line bundles on a smooth curve is a group with operation $\otimes$, inverses given by $(-)^\vee$, and identity given by $\mathcal{O}$. Use this to explain why bijection between the set of isomorphism classes of line bundles and the picard group is actually a group isomorphism.

5. Show that if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are locally free and

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is exact, then

$$0 \to \mathcal{F}_3^\vee \to \mathcal{F}_2^\vee \to \mathcal{F}_1^\vee \to 0$$

is also exact.

13 Thirteenth Set

1. **Proofs of the degree-genus formula.** Let $C \subset \mathbb{P}^2$ be a curve of degree d. Prove the degree genus formula...

   (a) using Hilbert Polynomials (exercise 6.7.3 in Gathmann): Compute the Hilbert polynomial of a hypersurface of degree d in $\mathbb{P}^n$ and use it to figure out the genus.

   (b) using Riemann-Hurwitz: Assume that C is *smooth* and does not contain the point $p = [0 : 0 : 1]$, and project $\mathbb{P}^2$ away from p onto $\mathbb{P}^1$. This induces a morphism from $C \to \mathbb{P}^1$. Use bezout's theorem to figure out the branching of the map, and then use Riemann-Hurwitz to compute the genus of C.

   (c) using a long exact sequence in sheaf cohomology: Explain where the following exact sequence comes from

   $$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0$$

   (here $i : C \to \mathbb{P}^2$ is the embedding). Explain why $h^k(\mathcal{O}_C) = h^k(i^*\mathcal{O}_C)$, and then use the long exact sequence in cohomology to figure out $g(C) = h^1(\mathcal{O}_C)$.

2. Let X be a smooth projective curve of genus 0. Show that if $\mathcal{L}_1, \mathcal{L}_2$ are line bundles of the same degree on X, then $h^0(\mathcal{L}_1) = h^0(\mathcal{L}_2)$.

3. Show that every smooth curve of genus zero is isomorphic to $\mathbb{P}^1$.

4. In $\mathbb{P}^1$, what is the relation between $\mathcal{O}(1)$ and $\mathcal{O}(p)$? What about on an elliptic curve?

5. Supply all the details from example 7.5.5.