Complex Square Roots

Enrique Acosta

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1 The difference between $\sqrt{(z-a)(z-b)}$ and $\sqrt{z-a}\sqrt{z-b}$

Let $a, b \in \mathbb{R}$ with $a < b$, and let $\sqrt{\bullet}$ be the square root in the complex plane with branch cut along $\mathbb{R}_{<0}$. How different are the complex functions $\sqrt{(z-a)(z-b)}$ and $\sqrt{z-a}\sqrt{z-b}$, where by $\sqrt{(z-a)(z-b)}$ we mean a composition? We remark that this interpretation on $\sqrt{(z-a)(z-b)}$ is certainly non-standard, and in this notes we will understand why.

To figure this out, we need to figure out when the expressions inside the root give values in the branch cut of $\sqrt{\bullet}$.

Now, $z - a \in \mathbb{R}_{<0}$ if and only if $Im(z) = 0$ and $Re(z) < a$ so the domain of definition of $\sqrt{z-a}$ is $\mathbb{C} - \mathbb{R}_{<a}$. Likewise $\sqrt{z-b}$ is defined only on $\mathbb{C} - \mathbb{R}_{<b}$. Therefore, the domain of definition of $\sqrt{z-a}\sqrt{z+a}$ is $\mathbb{C} - \mathbb{R}_{<b}$. Note that along the interval $\mathbb{R}_{(a,b)}$ the function $\sqrt{z-b}$ has a jump discontinuity given by a change of sign, while the function $\sqrt{z-a}$ is analytic. Therefore $\sqrt{z-a}\sqrt{z-b}$ has a sign change when you cross the interval $\mathbb{R}_{(a,b)}$. However , on the interval $\mathbb{R}_{<a}$ both $\sqrt{z-a}$ and $\sqrt{z-b}$ have sign changes, and so and so there is no sign jump at all! Thus, $\sqrt{z-a}\sqrt{z-b}$ can be extended analytically to $\mathbb{R}_{<a}$, and so we can actually make sense of the function $\sqrt{z-a}\sqrt{z-b}$ in $\mathbb{C} - \mathbb{R}_{(a,b)}$. To explicitly compute $\sqrt{z-a}\sqrt{z-b}$ on $\mathbb{R}_{<a}$ one just takes the two roots with positive imaginary part (and so in particular the output is a negative number).

Figure 1: Branch cut of $\sqrt{z-a}\sqrt{z-b}$

Now, for the domain of $\sqrt{(z-a)(z-b)}$ we need to figure out when $(z-a)(z-b) \in \mathbb{R}_{<0}$. Writing $z = x + iy$ we find this is equivalent to

$$\begin{aligned} iy(2x-a-b) &= 0 \\ x^2 - y^2 - ax - bx + ab &< 0. \end{aligned}$$

If $y = 0$, the solution is given by $(x-a)(x-b) < 0$ and so $\mathbb{R}_{(a,b)}$, and if $x = (a+b)/2$, the solution is $(a+b)/2 + i\mathbb{R}$. Thus the domain of $\sqrt{(z-a)(z-b)}$ is $\mathbb{C} - \big(\mathbb{R}_{(a,b)} \cup (a+b)/2 + i\mathbb{R}\big)$. Moreover, one can check that the branch along the line $(a+b)/2 + i\mathbb{R}$ is not removable because there is a sign change when one crosses the imaginary axis! For example, if $b = -a$ and $z = \delta + i$, then $z^2 - a^2 = \delta^2 - 1 - a^2 + i2\delta$ and as we change δ from positive to negative there is a sign change in the square root $\sqrt{\delta^2 - 1 - a^2 + i2\delta}$ because we cross the branch cut of $\sqrt{\bullet}$.

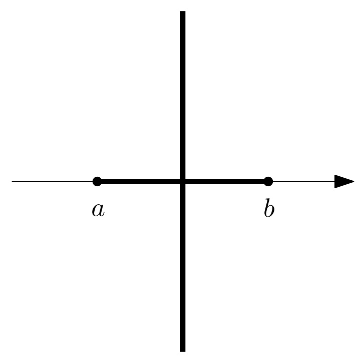

Figure 2: Branch cut of $\sqrt{(z-a)(z-b)}$

This, in particular, implies that $\sqrt{(z-a)(z-b)} \neq \sqrt{z-a}\sqrt{z-b}$ since $\sqrt{z-a}\sqrt{z-b}$ is analytic along the line $(a+b)/2 + i\mathbb{R}$. In fact, we have that outside $\mathbb{R}_{(a,b)}$

$$\sqrt{(z-a)(z-b)} = \begin{cases} \sqrt{z-a}\sqrt{z-b} & \text{if } Re(z) > (a+b)/2 \\ -\sqrt{z-a}\sqrt{z-b} & \text{if } Re(z) < (a+b)/2 \end{cases}$$

because of the uniqueness of analytic continuation and the fact that $\sqrt{(z-a)(z-b)} = \sqrt{z-a}\sqrt{z-b}$ on $\mathbb{R}_{>b}$ and $\sqrt{(z-a)(z-b)} = -\sqrt{z-a}\sqrt{z-b}$ on $\mathbb{R}_{<a}$ (for example, take $z = a - 1$ to find $\sqrt{(z-a)(z-b)} = \sqrt{b-a+1} \in \mathbb{R}$ and $\sqrt{z-a}\sqrt{z-b} = \sqrt{-1}\sqrt{a-b-1} = -\sqrt{b-a+1}$ by taking both roots with positive imaginary part as explained above).

Note, the above in particular shows (for $a = b = 0$) that

$$\sqrt{z^2} = \begin{cases} z & \text{if } Re(z) > 0 \\ -z & \text{if } Re(z) < 0 \end{cases}$$

which is easy to check directly.

2 The Riemann Surface of $\sqrt{z-a}\sqrt{z-b}$

The above is the reason why one has to be vary careful when interpreting expressions of the form $\sqrt{(z-a)(z-b)}$, since viewing them as compositions adds extraneous branch cuts. What is even more puzzling is that at any z one can clearly find two complex w which square to $(z-a)(z-b)$ (counting with multiplicities!), so it should be fine to evaluate the function on all the complex plane. The issue however is that if one starts at z_0 and choses one w_0 that squares to $(z_0-a)(z_0-b)$, and then continues analytically $w = \sqrt{(z-a)(z-b)}$ while going around $z = a$ and not around $z = b$, then when one comes back to z_0 the "function" is now returning $-w_0$, the other value!

One can show that this will not happen if one does not go around any of the points, or both points at the same time[1], and so for example there is a well defined analytic function on $\mathbb{C} - \mathbb{R}_{(a,b)}$ given by $\sqrt{(z-a)(z-b)}$ which returns w_0 at z_0. It is important to stress that here the expression $\sqrt{(z-a)(z-b)}$ is not a composition but instead a name we give to the function. The other square root will be then give by the negative of the function given by $-\sqrt{(z-a)(z-b)}$. For example, if $z_0 \in \mathbb{R}_{>b}$, and if we chose w_0 to be the negative root, and then we will have $\sqrt{(z-a)(z-b)} = -\sqrt{z-a}\sqrt{z-b}$, where the square roots are principal branches which is the function we studied in the previous section. If we chose w_0 to be the positive root, then $\sqrt{(z-a)(z-b)} = \sqrt{z-a}\sqrt{z-b}$.

Riemann's way to remedy this was to construct a Riemann surface with two sheets $\mathbb{C} - \mathbb{R}_{(a,b)}$, and then glue them correctly. Here what one does is to use the *symmetry principle*, which states that if one has two holomorphic functions on two regions whose boundaries share an interval, and so that the continuous extensions of both functions agree on the interval, then one can glue them to get a holomorphic function![2]

In our case, and for definiteness, let

$$f_1(z) = \sqrt{(z-a)(z-b)} = \sqrt{z-a}\sqrt{z-b}$$

on one sheet, and let

$$f_2(z) = -\sqrt{(z-a)(z-b)} = -\sqrt{z-a}\sqrt{z-b}$$

on the other sheet. Note that as you approach the cut $\mathbb{R}_{(a,b)}$ form above, $f_1(z)$ extends to a function that returns the positive imaginary root, while as you approach the cut from below $f_1(z)$ extends to a function that returns the negative imaginary root. Because of this, if we want to extend $f_1(z)$ we should really see the cut as two intervals, and a nice way to do this is to enlarge the slit in $\mathbb{C} - \mathbb{R}_{(a,b)}$ so that it becomes a circle by a conformal map. The precise conformal map that does this is well known (see [1]) and given by

$$z = \frac{b-a}{4}\left(w + \frac{1}{w}\right) + \frac{a+b}{2}.$$

The map

$$\phi : \mathbb{C}_w \to \mathbb{C}_z$$

[1] See *Lectures on the Theory of Elliptic Functions,* by Harris Hancock, art. 114 p 133.

[2] See for example Stein and Shakarchi p. 60. The statement there is only for regions symmetric with respect to the real axis, but the proof works in general.

is map is 2 to 1 and the preimage of any $z \notin \mathbb{R}_{[a,b]}$ is given by a point w_1 inside the unit w-circle and a point w_2 outside of it, with $w_1 w_2 = 1$. The preimages of the $z \in \mathbb{R}_{(a,b)}$ are two points, one in the upper half of the w-circle, one in the lower half with equal real parts. Finally, ϕ is branched along the two points $w = -1$, $w = 1$ which are the only preimages of the points $z = a$ and $z = b$ respectively. We therefore see that ϕ maps the complement U of the closed unit disc in the w-plane centered at the origin conformally and bijectively to $\mathbb{C} - \mathbb{R}_{(a,b)}$, and it also maps the interior B of the disc conformally and bijectively to $\mathbb{C} - \mathbb{R}_{(a,b)}$.

Then we can define f_1 on U by using the conformal map (we will still call it f_1) and now we can extend $f_1(w)$ continuously to the boundary $\partial U = S^1$. By what was discussed above, this extension gives positive purely imaginary values on the top part of the circle, while it gives negative imaginary values on the bottom part of the circle, corresponding to the two extensions of $f_1(z)$ to the slit while approaching from above or from below.

We can also define f_2 on B by using the conformal map, and by the same reasoning, $f_2(w)$ will have a continuous extension to $\partial B = S^1$. If one analyses the conformal map carefully, one sees that approaching the top part of S^1 form B corresponds to approaching the slit from below in the z-plane, and so the extension of $f_2(w)$ to the top part of S^1 corresponds to the extension of $f_2(z)$ to the slit when approaching from below, which will give the positive purely imaginary root, since $f_2(z) = -f_1(z)$.

This shows that the extensions of $f_1(w)$ and $f_2(w)$ agree on S_1, and so by the symmetry principle we mentioned above we can glue them to obtain an analytic function $\mathcal{F}$ on the whole w-plane (it will have a pole at $w = \infty$)! This w plane is the Riemann surface of $\sqrt{z-a}\sqrt{z-b}$, which now has no branch cuts. Explicitly, it is given by

$$\begin{aligned}
\mathcal{F}(w) &= \begin{cases} f_1(\frac{b-a}{4}\left(w + \frac{1}{w}\right) + \frac{a+b}{2}) & \text{if } w \in U \\ f_2(\frac{b-a}{4}\left(w + \frac{1}{w}\right) + \frac{a+b}{2}) & \text{if } w \in B \end{cases} \\
&= \begin{cases} f_1(\frac{b-a}{4}\left(w + \frac{1}{w}\right) + \frac{a+b}{2}) & \text{if } w \in U \\ -f_1(\frac{b-a}{4}\left(w + \frac{1}{w}\right) + \frac{a+b}{2}) & \text{if } w \in B \end{cases}
\end{aligned}$$

and the extension from above of f_1 in the top part of S^1 and the extension from below of f_1 on the bottom part of S^1, which in this case will be positive purely imaginary values on the top part of S^1 and negative purely imaginary values on the top part of S^1.

References

[1] M.R. Spiegel. *Schaum's outlines: complex variables: with an introduction to conformal mapping and its applications.* Schaum's Outline Series, 1964.