Intersection Theory on Smooth Surfaces

Enrique Acosta

September 2011

Surface: An irreducible, smooth algebraic surface in some projective space over $\mathbb{C}$, or some open subset of such. (Hartshorne: Any complete (i.e. universally closed, the equivalent to compact according to Mumford Red Book I.9) nonsingular algebraic variety of dim 2 over $\mathbb{C}$ is projective...smoothness is important!)

Curve on a surface S: A closed, reduced, irreducible 1-dimensional subvariety of S.

Divisor on S: An expression of the form

$$\sum_{C \text{ curve in } S} n_c C$$

where $n_c \in \mathbb{Z}$ are almost all zero.

The divisor coming from a rational function: $f \in K(S) - 0$**:**

$$\operatorname{div}(f) = \sum_{C \text{ curve in } S} \operatorname{ord}_C(f) C$$

where $\operatorname{ord}_C(f)$ is the valuation of f in the discrete valuation ring $\mathcal{O}_{S,C}$.

Analogy: Think of functions of a complex variable. If $f(z) = z^3/(z-i)^2$, then we would write

$$\operatorname{div}(f) = 3(0) - 2(i).$$

What the valuation generalizes if the following: For any meromorphic function $f(z)$ and any a we have

$$f(z) = (z-a)^n g(z)$$

where $n \in \mathbb{Z}$ and $g(z)$ is analytic and non-zero. Then $\operatorname{ord}_a(f) = n$. Note: $z-a$ is the generator of the maximal ideal in the local ring, and $g(z)$ is a unit.

Example: In $\mathbb{A}^2_{x.y}$ we have that $f = x^2/y$ has $\operatorname{div}(f) = 2L_x - L_y$ where L_x is the line $x = 0$ and L_y is the line $y = 0$.

Principal Divisors: Divisors of the form div (f) are called *principal.* They for a subgroup of the group $\mathrm{Div}(S)$ of all divisors because $\mathrm{div}\,(f_1 f_2) = \mathrm{div}\,(f_1) + \mathrm{div}\,(f_2)$.

Note: Every divisor is locally principal! (because of smoothness). But not every divisor is principal.

EXAMPLE: Let L be the line $x_0 = 0$ in $\mathbb{P}^2_{x_0,x_1,x_2}$. Then L is not principal because x_0 is not a function on $\mathbb{P}^2$! However, on the opens $U_i = \{x_i \neq 0\}$ we have

$$L \mid_{U_i} = \mathrm{div}\left(\frac{x_0}{x_i}\right).$$

Linear Equivalence Two divisors D_1 and D_2 are said to be *linearly equivalent* if there is an f with

$$\mathrm{div}\,(f) = D_1 - D_2$$
$$\left(\mathrm{div}\left(\frac{1}{f}\right) = D_2 - D_1\right)$$

EXAMPLE: Any curve of degree d in $\mathbb{P}^2$ is linearly equivalent to dL where L is a line.
Reason. Let $C = Z(F)$ where F is a homogeneous polynomial of degree d, and let $L = Z(x_o)$. Note that $F \notin K(S)$, but F/x_0^d is! Moreover,

$$\mathrm{div}\left(\frac{F}{x_0^d}\right) = C - dL.$$

EXAMPLE: Any two curves in $\mathbb{A}^2$ are linearly equivalent.
(if $C_i = Z(f_i)$, then $C_1 - C_2 = \mathrm{div}\left(\frac{f_1}{f_2}\right)$)

Picard group of S: $\mathrm{Pic}(S) = \mathrm{Div}(S)/\{\text{principal divisors}\}$

EXAMPLES:

- $\mathrm{Pic}(\mathbb{A}^2) = 0$
- $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}[L]$ where L is the class of a line.
- If S' is the blow-up of S at a point, then $\mathrm{Pic}(S') = \mathrm{Pic}(S) \oplus \mathbb{Z}[E]$ where E is the exceptional divisor.

Intersections of Divisors: Let D_1 and D_2 be divisors on S. We say that D_1 and D_2 are in *general position* if their supports have no common components where

$$\text{Supp}(\sum n_c C) = \bigcup\{C \mid n_c \neq 0\}.$$

FACT: If this is the case, then $\text{Supp}(D_1) \cap \text{Supp}(D_2)$ will consist of finitely many points (being closed and of dim 0).

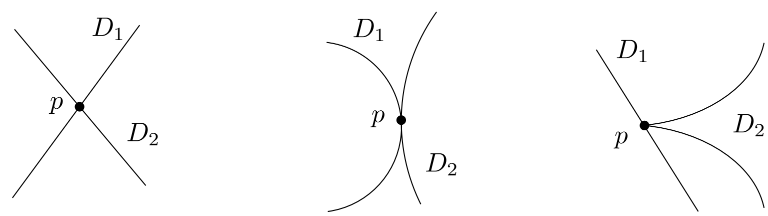
Say, D_1, D_2 are effective (≥ 0) and in general position. We define

$$D_1 \cdot D_2 = \sum_{p \in \text{Supp}(D_1) \cap \text{Supp}(D_2)} (D_1, D_2)_p$$

where

$$(D_1, D_2)_p = \dim_k \left(\mathcal{O}_{S,p}/(f_1, f_2)\right)$$

where the f_i are local equations for the D_i around p. This is called the *local intersection multiplicity at* p.



$(D_1, D_2)_p = 1$ $\qquad$ $(D_1, D_2)_p > 1$ $\qquad$ $(D_1, D_2)_p > 1$

We extend this definition to non-effective divisors by bilinearity (any divisor can be written on the form $D = D' - D''$ with $D', D'' \geq 0$).

Then we extend it to divisors that are not in general position by defining

$$D_1 \cdot D_2 = D_1' \cdot D_2'$$

where D_i' is linearly equivalent to D_i and D_1', D_2' are in general position.

This is well defined, but it is painful to check, and so we end up with a product which

- is bilinear and symmetric
- is independent of linear equivalence representatives.

For all the details, see Shararevich's Basic Algebraic Geometry chapter IV.

Note for the experts There is a slick way to define this intersection by noting that for any two line bundles $\mathcal{F}_1, \mathcal{F}_2$ on S, the expression

$$(\mathcal{F}_1, \mathcal{F}_2) = \chi(\mathcal{O}_S) - \chi(\mathcal{F}_1^{-1}) - \chi(\mathcal{F}_2^{-1}) + \chi(\mathcal{F}_1 \otimes \mathcal{F}_2)$$

is bilinear (with $\otimes$ playing the role of the sum!). One applies this with $\mathcal{F}_1 = \mathcal{O}_S(D_1)$ and $\mathcal{F}_2 = \mathcal{O}_S(D_2)$, and then it al boils down to proving that if C and D are two irreducible curves in general position, then

$$(\mathcal{O}_S(C), \mathcal{O}_S(D)) = C \cdot D.$$

This is proved using the fact that in this case there exact sequence of sheaves

$$0 \to \mathcal{O}_S(-C-D) \to \mathcal{O}_S(C) \oplus \mathcal{O}_S(D) \to \mathcal{O}_S \to \mathcal{O}_{C\cap D} \to 0$$

(note that $\mathcal{O}_{C\cap D} = \mathcal{O}_S/(\mathcal{O}_S(-C) + \mathcal{O}_S(-D)))$, and the fact that the stalk of $\mathcal{O}_{C\cap D}$ at a point of their intersection is precisely $\mathcal{O}_{S,p}/(f_1, f_2)$ which is what we used to define the local intersection number $(C, D)_p$ above.

Note: This approach takes care of the independence of the intersection with linear equivalence because if D and D' are linearly equivalent, then $\mathcal{O}_S(D) = \mathcal{O}_S(D')$!

For a details see Beauville's Complex Algebraic Surfaces.

Example Let C_i be any two curve of degree d_i in $\mathbb{P}^2$. Then

$$\begin{aligned} C_1 \cdot C_2 &= (d_1 L)\cdot(d_2 L) \\ \text{(bilinearity)} &= (d_1 d_2) L \cdot L \\ (L' \text{ some other line}) &= (d_1 d_2) L \cdot L' \\ &= d_1 d_2 \end{aligned}$$

which is Bezout's theorem!

Example A negative self intersection:

Let S be a smooth surface of degree d in $\mathbb{P}^3$ defined by $F = 0$, and let L be a line contained in S.

Let E be the intersection of a plane $\mathbb{P}^2$ containing L with S. Then

$$E = L + C$$

where C is some curve in $\mathbb{P}^2$ of degree $d-1$. Multiplying by L we get

$$\begin{aligned} E \cdot L &= (L + C)\cdot L \\ &= L \cdot L + C \cdot L \end{aligned}$$

but $E \cdot L = 1$ because one can move the plane that cuts E so that it does not contain L, and then the intersection of the plane and the line is just one point. Also, $C \cdot L = d - 1$ since C has degree $d - 1$. This shows that

$$L^2 = L \cdot L = 2 - d$$

which is negative if $d \geq 3$!

For example, any line on a smooth cubic surface in $\mathbb{P}^3$ is a -1-curve.

Pull-back of a divisor Let S, S' be smooth surfaces, D be a divisor of S, and $\pi : S' \to S$ a morphism. We define $\pi^* D$, the pullback of D to S' as follows:

For any C in the support of D, let f be a local equation. Define

$$\pi^* C = \operatorname{div}(\pi^* f) = \operatorname{div}(f \circ \pi)$$

and extend this to D by linearity.

Note: There is an issue here with f being only local, and what should be understood here from the above equation, is that it is the way to find the coefficients of the components of $\pi^{-1}(C) = \operatorname{Supp} \pi^* C$.

Example: The self intersection of and exceptional divisor:

Let S' be the blow-up of S at a point p, and let E be the exceptional divisor (a curve). Let C be a smooth curve going through p with multiplicity 1 (i.e. smooth at p). We have

$$\pi^* C = C' + E$$

where C' is the strict transform of C. Thus, multiplying by E we get

$$E \cdot \pi^* C = E \cdot C' + E^2,$$

but $E \cdot C' = 1$ (the point corresponding to the tangent line of C at p), and

$$E \cdot \pi^* C = 0$$

because we can move C away from p (this requires S to be smooth...one can always move a divisor away from finitely many points. Shafarevich III 1.3 Theorem 1) to say $C \equiv D$, and then

$$E \cdot \pi^* C = E \cdot \pi * D = 0$$

because $\pi^* C \equiv \pi^* D$ and $\pi^* D$ does not intersect E. This implies that

$$E^2 = -1.$$

Some comments on negative self intersections When we have an effective curve with $C^2 < 0$, this in particular implies that we cannot "move" the curve in the sense that there is no effective (≥ 0) divisor that is equivalent to it. This is because if we could find a divisor $D \geq 0$ with $C \equiv D$ and $C \notin \mathrm{Supp}(D)$, then $C^2 = C \cdot D \geq 0$ since $C \geq 0$ too by bilinearity.

Combining this with the statement from Shafarevich that we used above stating that one can always move a divisor away from finitely many points (Shafarevich III 1.3 Theorem 1), we see that if $C^2 < 0$, then even though we can always move C away from a point, every time we do so we need to use negative coefficients, and moreover that any D with $C \equiv D$ not only has negative coefficients, but it also always has to intersect C.

A cool theorem of Castelnuovo A curve E on a smooth surface S' can be contracted to point on a smooth surface S (ie, a map $\pi : S' \to S$ with S smooth and $\pi : S' - E \xrightarrow{\cong} S - \{p\}$) if and only if $E \cong \mathbb{P}^1$ and $E^2 = -1$.

Example: Hirzebruch surfaces:

$\mathbb{F}_n$ is the surface in $\mathbb{P}^2_x \times \mathbb{P}^1_t$ defined by $x_1 t_1^n = x_2 t_2^n$. They are all ruled surfaces

$$\begin{array}{c} \mathbb{F}_n \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

(the fibers are all isomorphic to $\mathbb{P}^1$). Moreover, any ruled surface over $\mathbb{P}^1$ is isomorphic to one of these because

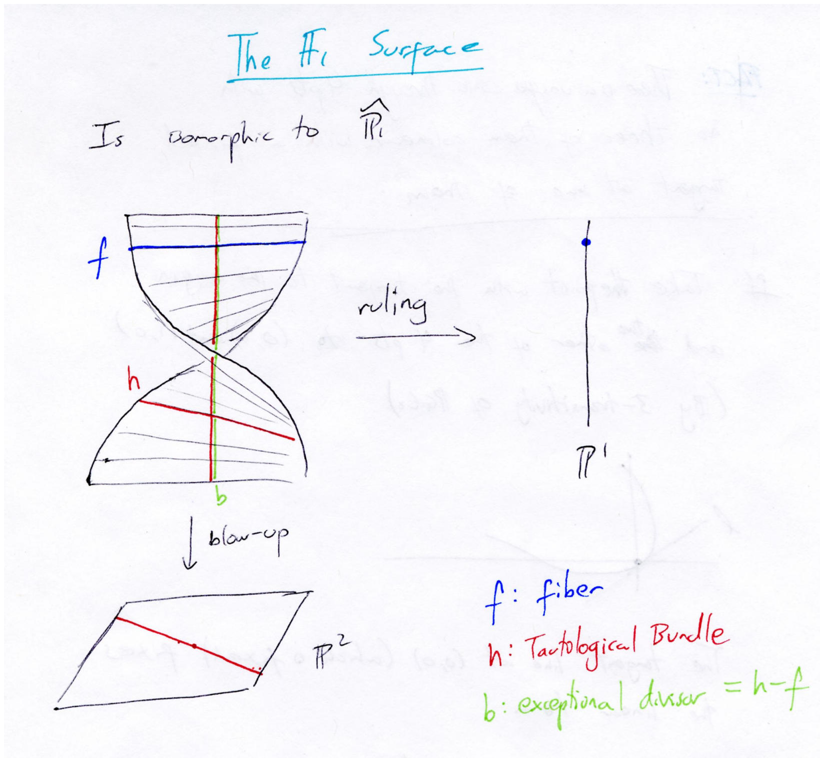
$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

and using the fact that we know how rank 2 vector bundles over $\mathbb{P}^1$ look like.

- $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$
- $\mathbb{F}_1 = Bl_{pt}\mathbb{P}^2$
- $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}[B] \oplus \mathbb{Z}[F]$ where F is the class of a fiber, and B is the class of the unique irreducible curve in $\mathbb{F}_n$ with negative self intersection.
- $F^2 = 0$, $B^2 = -n$, $B \cdot F = 1$

In the picture H is the pullback of a line. One has

$$H = B + F$$



The Canonical Divisor K_S=the canonical divisor on S= the divisor of a 2-form on S.

This is well defined only up to linear equivalence!

Example: Let $S = \mathbb{P}^2_{X,Y,Z}$ and $U = \mathbb{A}^2_{x,y} = \{Z \neq 0\}$ where $x = X/Z, y = Y/Z$. Let ω be the 2-form that on U is give by

$$\omega = dx \wedge dy.$$

Then $\operatorname{div}(\omega) \mid_U = 0$. To look at what is the coefficient of $L = \{Z = 0\}$ go to another chart, say $\{Y \neq 0\}$ with coordinates $u = X/Y = x/y$ and $v = Z/Y = 1/y$ where L has equation $v = 0$. Then

$$\omega = d\left(\frac{u}{v}\right) \wedge d\left(\frac{1}{v}\right) = -\frac{1}{v^3} du \wedge dv$$

and so $\operatorname{ord}_H(\omega) = -3$ which implies that

$$K_{\mathbb{P}^2} = -3L.$$

The adjunction formula If C is a smooth curve on a surface S, then the differentials of C are related to the differential 2-forms on S by (ω_S: the canonical sheaf on S)

$$(\omega_S \otimes \mathcal{O}_S(C)) \mid_C = \mathcal{O}_C(K_C)$$

which we may write as

$$\mathcal{O}_S(K_S + C) \mid_C = \Omega^1_C$$

since $\omega_S = \mathcal{O}(K_S)$ and $\Omega^1_C = \mathcal{O}_C(K_C)$.

Taking degrees this gives another (weaker) form of adjunction:

$$K_S \cdot C + C^2 = 2g(C) - 2$$

For C not necessarily smooth, one has

$$K_S \cdot C + C^2 = K_S \cdot D + D^2 = 2g(D) - 2$$

where D is a smooth curve that is linearly equivalent to C. As long as we know the existence of such a C, we can conclude that

$$K_S \cdot C + C^2 \geq -2,$$

which we will use later on. However, by Hartshorne V.ex.1.3 and III.ex.5.3 we do not need to find a smooth D equivalent to C to show that

$$K_S \cdot C + C^2 \geq -2$$

independent of whether C is smooth or not.

If C is not smooth, one can also prove that that

$$K_S \cdot C + C^2 = (2g(\widetilde{C}) - 2) + \sum r_i(r_i - 1)$$

where $g(\widetilde{C})$ is the geometric genus of C (the genus of of its desingularization, or normalization), and the r_i are the multiplicities of all the infinitely near points lying over the singular points of C. We will discuss and explain this in detail below.

Example: Let L be the class of a line in $\mathbb{P}^2$. Then using adjunction with $C = L \cong \mathbb{P}^1$ we get

$$K_{\mathbb{P}^3} \cdot L + 1 = -2$$

and so $K_{\mathbb{P}^2} \cdot L = -3$. But $K_S = aL$ for some a since $\text{Pic}(\mathbb{P}^2) = \mathbb{Z}[L]$, and so

$$K_{\mathbb{P}^2} = -3L.$$

Example: Let C be a smooth curve of degree d in $\mathbb{P}^2$. Then $C \equiv dL$ and using adjunction we get

$$-3L \cdot dL + d^2 L \cdot L = 2g(C) - 2$$

and so

$$g(C) = \frac{(d-1)(d-2)}{2}$$

the degree genus formula!

Invariance under pullbacks Let $f : S' \to S$ be a generically finite morphism of degree d, and let D_1, D_2 be divisors in . Then

$$f^*D_1 \cdot f^*D_2 = d(D_1 \cdot D_2).$$

For the proof the idea is to move D_1 and D_2 so that they intersect transversally and their intersections occur at points of S where there is no branching of f, so there are d distinct preimages. Then f^*D_1 and f^*D_2 intersect transversally at each of these preimages and so the number of intersection points gets multiplied by d. The fact that you can always move the D_i like this is a little delicate and uses a big theorem by Serre (see Beauville p.4.)

Example The Veronese surface in $\mathbb{P}^5$ contains no lines:

The Veronese surface S is the image of $\sigma : \mathbb{P}^2 \to \mathbb{P}^5$ by the linear system of conics, and so its hyperplane sections H are conics all conics. Note that all hyperplane sections are linearly equivalent since we can move the plane. If L were a line contained in S, then

$$1 = H \cdot L$$

because we can move the hyperplane, and any hyperplane that does not contain L intersects it at one point. This makes no sense because if we pull this back to $\mathbb{P}^2$, then

$$1 = \sigma^*H \cdot \sigma^*L = 2l \cdot \sigma^*L = 2(l \cdot \sigma^*L)$$

where l is the class of a line in $\mathbb{P}^2$.

The canonical divisor of a Blow-up If $S' \xrightarrow{\pi} S$ is the blow-up of S at a point, then

$$K_{S'} = \pi^*K_S + E.$$

This can be proved with local equations (see Bueauville p. 13).

Resolution of Singularities Let C be a curve which has a singularity at p of order r (if f is a local equation of C around p, then $f \in \mathfrak{m}_{S,p}^r$ but $f \notin \mathfrak{m}_{S,p}^{r+1}$), and let $S' \xrightarrow{\pi} S$ be the blow up of S at p. Then one can prove (again, using local equations) that

$$\pi^*C = C' + rE$$

where C' is the strict transform of C and E is the exceptional curve.

Moreover, one has

$$\begin{aligned} C' \cdot E &= (\pi^*C - rE) \cdot E \\ &= \pi^*C \cdot E + r \\ &= r \end{aligned}$$

since we can move C away from p (because of smoothness of S!).

But then using $K_{S'} = \pi^* K_S + E$ one has

$$\begin{aligned} C' \cdot K_{S'} &= C' \cdot (\pi^* K_S + E) \\ &= C' \cdot \pi^* K_S + r \end{aligned}$$

and

$$\begin{aligned} C' \cdot C' &= (\pi^* C - rE) \cdot (\pi^* C - rE) \\ &= \pi^* C \cdot \pi^* C - 2r\pi^* C \cdot E - r^2 \\ &= C \cdot C - r^2 \end{aligned}$$

where the equality $\pi^* C \cdot \pi^* C = C \cdot C$ follows from the fact that π is generically of degree 1. We conclude that

$$C' \cdot C' + C' \cdot K_{S'} = C \cdot C + C \cdot K_S - r(r-1).$$

This proves that the quantity

$$C^2 + C \cdot K$$

is decreasing on the strict transforms of the curve as we blow up the singular points of C on the surface, but by adjunction (the strong version that we can use on singular curves) we always have

$$C^2 + C \cdot K \geq -2.$$

Thus, the quantities $C^2 + C \cdot K$ must eventually stabilize, and so the r's must eventually all become 1. This just means that eventually the curve is smooth!

(taken from Shafarevich's book on surfaces in the Encyclopedia of Mathematics series). See also Hartshorne V.3.7 and 3.8.

The genus of the desingularization The proof above gives the extra statement: Let $\widetilde{C}$ be the smooth curve you obtain at the end of the above process, $\widetilde{S}$ be the smooth surface where it lives, and let r_i be the multiplicities of all the points that got blown up in the resolution (these points lie on intermediate surfaces, and project down to the singular points of C). Then we have

$$C^2 + K_S \cdot C = \widetilde{C}^2 + K_{\widetilde{S}} \cdot \widetilde{C} + \sum r_i(r_i - 1)$$

(we had referred before to these r_i as the multiplicities of all the infinitely near points lying over the singular points of C). Now, by adjunction on $\widetilde{S}$ and by the smoothness of $\widetilde{S}$ we obtain

$$C^2 + K_S \cdot C = 2g(\widetilde{C}) - 2 + \sum r_i(r_i - 1)$$

from which you can get the genus of the desingularization.

Note that to find the r_i you have to go through the whole desingularization process. In the case where the singular points of C are *ordinary* (a point p of multiplicity r is ordinary if C has r distinct tangent lies at p), then the sum ranges only over the multiplicities of the singular points of C.

The general degree-genus formula If we start with a plane curve C of degree d, then we know that $C^2 + K_S \cdot C = (d-1)(d-2)/2$ and using the above formula we find that the geometric genus of a curve of degree d in $\mathbb{P}^2$ is given by

$$g = \frac{(d-1)(d-2)}{2} - \sum \frac{r_i(r_i-1)}{2}.$$

This is the most general form of the degree-genus formula for plane curves.

Again, in the case when the singularities are ordinary, the only r_i that show up are the multiplicities of the singular points of C. For conics all singularities are ordinary, and for cubics only nodes are.