The number of lines intersecting four lines in 3-space

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There are exactly 2 lines intersecting four sufficiently general lines in 3-space. In this project you will prove this statement, and in the process understand precisely what sufficiently general means, and learn about 3-dimensional projective space, which is where the statement is true. This is one of the simplest and nicest counts in the field of *Enumerative Geometry*, part of Algebraic Geometry.

1 Projective Spaces

The n-dimensional projective space $\mathbb{RP}^n$ over the real numbers is defined the be the set $\mathbb{R}^{n+1} - \{(0,\dots,0)\}$ modulo the equivalence relation $(x_0, x_1, \dots, x_n) \sim (\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ for $\lambda \in \mathbb{R}^\times$. We denote the points in this space using projective coordinates $[x_0 : x_1 : \dots : x_n]$, which represent the equivalence class of the point $(x_0, x_1, \dots, x_n)$, where of course $x_0, x_1, \dots, x_n$ are not all zero. Note that $\mathbb{RP}^n$ can be naturally identified with the set of lines in $\mathbb{R}^{n+1}$ through the origin.

One important property we will be using often about the projective space $\mathbb{RP}^n$ is that it is covered by some natural copies of $\mathbb{R}^n$. Explicitly, the sets U_i for $i = 0, 1, \dots, n$ of points of $\mathbb{RP}^n$ with projective coordinate $x_i \neq 0$ are in natural bijection with $\mathbb{R}^n$. We call these U_i *affine charts.* The explicit bijection is given by

$$\begin{array}{ccc} U_i & \to & \mathbb{R}^n \\ [x_0 : x_1 : \dots : x_n] & \mapsto & \left(\dfrac{x_0}{x_i}, \dfrac{x_1}{x_i}, \dots, \widehat{\dfrac{x_i}{x_i}}, \dots, \dfrac{x_n}{x_i}\right). \end{array}$$

Exercise 1. Prove this statement, and show that the sets U_i are open with the quotient topology on $\mathbb{RP}^n$ coming from $\mathbb{R}^{n+1}$. (Optional) Moreover, show that the map above is a homeomorphism.

Exercise 2. Prove that the complement in $\mathbb{RP}^n$ of each U_i is naturally a copy of $\mathbb{RP}^{n-1}$. This is why we say that $\mathbb{RP}^1$ is $\mathbb{R}$ with an extra point "at infinity" (the extra point being $\mathbb{RP}^0$), and $\mathbb{RP}^2$ is $\mathbb{R}^2$ with an extra line "at infinity" (an $\mathbb{RP}^1$), and so on.

2 Lines in $\mathbb{RP}^3$

Let Π be a 2-dimensional subspace of $\mathbb{R}^4$. We call the image of $\Pi - \{(0,0,0,0)\}$ in the quotient space $\mathbb{RP}^3$ a *line* (remember that 1-dimensional subspaces of $\mathbb{R}^4$ correspond to points in $\mathbb{RP}^3$). For example, the line defined by $x_0 = x_1 = 0$ is the set of points of the form $[0 : 0 : x_2 : x_3]$ and is clearly isomorphic to $\mathbb{RP}^1$.

Similarly, we call the set of points coming from a 3-dimensional subspace of $\mathbb{R}^4$ a *plane.*

Exercise 3. Show that a line in $\mathbb{RP}^3$ that does not lie in the complement of U_0 does in fact look like a line in the usual sense when you identify U_0 with $\mathbb{R}^3$ with the map described above (use x, y, z as coordinates of $\mathbb{R}^3$ where $x = x_1/x_0$, $y = x_2/x_0$ and $z = x_3/x_0$). Moreover, show that *any* line in $\mathbb{R}^3$ comes from a line in $\mathbb{RP}^3$ when you identify $\mathbb{R}^3$ with U_0.

Exercise 4. A change of coordinates in $\mathbb{RP}^n$ is given by a map induced by a linear isomorphism $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ (a change of basis of the vector space $\mathbb{R}^{n+1}$). Explain why changes of coordinates take lines to lines, and why one can take any line in $\mathbb{RP}^3$ onto any other line by a linear change of coordinates. Conclude that they are all copies of $\mathbb{RP}^1$ lying inside $\mathbb{RP}^3$. Do the same for planes, and conclude they are all isomorphic to $\mathbb{RP}^2$.

Exercise 5. Show that if two lines in $\mathbb{RP}^3$ lie on a plane, then they necessarily intersect. In particular, conclude that any two lines in $\mathbb{RP}^2$ intersect. What is the intersection point of the lines $x = 1$ and $x = 2$ in $\mathbb{R}^2$ when viewed inside $\mathbb{RP}^2$?

Exercise 6. Show that any set of three non-intersecting lines L_1, L_2, L_3 in $\mathbb{RP}^3$ can be taken by a change of coordinates to any other set of non-intersecting lines.

Exercise 7. Let $p = [p_0 : p_1 : p_2 : p_3]$ and $p = [q_0 : q_1 : q_2 : q_3]$ be any two points in $\mathbb{RP}^3$. Show that there is a unique line in going through these two points, and that it is the image of the map

$$\begin{aligned} \mathbb{RP}^1 &\to \mathbb{RP}^3 \\ [s : t] &\mapsto [sp_0 + tq_0 : \ldots : sp_3 + tq_3]. \end{aligned}$$

We call this the *parametrization* of the line through the points p and q, and we informally write it as $sp + tq$ for $[s : t] \in \mathbb{RP}^1$.

3 The Segre Embedding

There is a very nice way to realize the cartesian product $\mathbb{RP}^1 \times \mathbb{RP}^1$ as a surface in $\mathbb{RP}^3$, and this fact and its generalizations are of great importance in algebraic geometry. The explicit map in this situation is defined by

$$\begin{aligned} \phi : \mathbb{RP}^1 \times \mathbb{RP}^1 &\to \mathbb{RP}^3 \\ [s_0 : s_1] \times [t_0 : t_1] &\mapsto [s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1] \end{aligned}$$

Exercise 8. Show that ϕ is a bijection.

Exercise 9. Show that the image $\mathcal{S}$ of ϕ is precisely the set of points of $\mathbb{RP}^3$ that satisfy the equation $x_0x_3 - x_1x_2 = 0$. We call this surface $\mathcal{S}$ a *quadric* because it is defined by a homogeneous equation of degree two. Draw a picture of how $\mathcal{S}$ looks like inside U_0.

Exercise 10. Show that the image of the sets $p \times \mathbb{RP}^1$ under ϕ are all lines. We say that the quadric surface $\mathcal{S}$ is ruled because the lines sweep the whole surface. Similarly, show that the image of the sets $\mathbb{RP}^1 \times q$ for varying q is another ruling of the quadric. Explain why any two lines of the same ruling do not intersect, and show that any two lines of different rulings intersect at precisely one point. Show these lines in the picture in U_0.

Exercise. (Optional) Show that by a change of coordinates on can take $\mathcal{S}$ to the surface defined by $y_0^2+y_1^2 = y_2^2+y_3^2$ in $\mathbb{RP}^3$ in the new projective coordinates $[y_0, y_1, y_2, y_3]$. Draw a picture of this transformed $\mathcal{S}$ in the new U_0. The two rulings of $\mathcal{S}$ are a little easier to see in the pictures you can draw by hand.

Exercise 11. Show that any line in $\mathbb{RP}^3$ either is contained in $\mathcal{S}$ or else it intersects it at 0,1 or 2 points. Explain why, if we allowed the variables to be in $\mathbb{C}$, and kept all the definitions the same, then the lines of $\mathbb{CP}^3$ would either be contained in the surface $\mathcal{S}$ or would intersect $\mathcal{S}$ at *exactly* two points counting multiplicities (Hint: use exercise 7).

The above is a very simple case of Bezout's theorem, which tells you the number of intersection points of two projective varieties you expect to intersect at finitely many points. Bezout's theorem is only true over algebraically closed fields, which is why you only get an upper bound in exercise 11 when working over $\mathbb{R}$.

We will need one more fact about the surface $\mathcal{S}$ that we cannot prove with elementary means because it relies on the structure of $\mathcal{S}$ as an algebraic variety, and in particular on properties of its tangent planes. We state the fact we need below.

Fact 12. *There are exactly two lines in $\mathcal{S}$ through any point p on $\mathcal{S}$.*

Note that by exercise 10 we know precisely which these two lines are! The fact states that there are no other lines in $\mathcal{S}$ through p.

Sketch of proof of fact 12. Any line contained in $\mathcal{S}$ going through a point p must be contained in the tangent plane to $\mathcal{S}$ at p. Thus, the intersection of the tangent plane to $\mathcal{S}$ at p with $\mathcal{S}$ contains the two lines of the two rulings going through p. But, any general plane in $\mathbb{RP}^3$ intersects $\mathcal{S}$ along a conic, and this implies that the intersection of the tangent plane with $\mathcal{S}$ must be precisely the union of the two lines since this is already a conic. This proves there are no more lines contained in $\mathcal{S}$ through p.

4 Four lines in space

Exercise 13. Show that given three non-intersecting lines L_1, L_2, L_3 in $\mathbb{RP}^3$, the union of the lines in $\mathbb{RP}^3$ that intersect the three lines is a quadric surface in $\mathbb{RP}^3$. Hint: Use a linear transformation to take the lines to three lines in one ruling of $\mathcal{S}$, and use exercises 10 and 11.

We now have all the required background to state and prove the statement!

Exercise 14. Prove that there are at most two lines intersecting a fixed set of 4 non-intersecting lines in $\mathbb{R}^3$ that do not all lie on a quadric surface by showing the statement for $\mathbb{RP}^3$. Hint: Use fact 12.

Exercise 15. Prove the nice exact statement that is only true over $\mathbb{C}$: There are *exactly* two lines (counting multiplicities) intersecting a fixed set of 4 non-intersecting lines in $\mathbb{CP}^3$ that do not all lie on a quadric surface. Explain what the "counting multiplicities" means precisely.