Rational Tetrahedra

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por

Enrique Acosta Jaramillo

Director: Alf Onshuus
Codirector: Ronald van Luijk

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A Xavier Caicedo.

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Introduction

In this thesis we will study some arithmetic problems related to the edges and volume of a tetrahedron. We will call a tetrahedron with rational edges and volume a *rational* tetrahedron. The equation relating the volume V of a tetrahedron and the lengths of its edges a, b, c, d, e, f is given by

$$\begin{aligned}(12V)^2 &= (a^2+d^2)(-a^2d^2+b^2e^2+c^2f^2)+\\ &\quad (b^2+e^2)(a^2d^2-b^2e^2+c^2f^2)+\\ &\quad (c^2+f^2)(a^2d^2+b^2e^2-c^2f^2)-\\ &\quad a^2b^2c^2-a^2e^2f^2-b^2d^2f^2-c^2d^2e^2,\end{aligned} \tag{1}$$

where the edges are arranged in the configuration shown below.

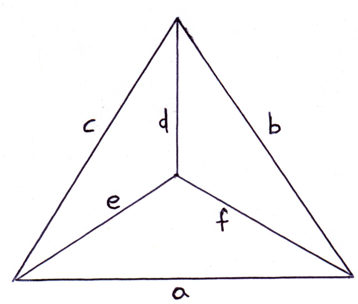

Figure 1. The General Tetrahedron

This equation has been rediscovered several times (see [Chi04] for references, and [Buc92] for a proof) and its solutions in integers or rational numbers have been studied extensively. We know that (1) does have integer solutions (for example $(a, b, c, d, e, f, V) = (7, 6, 4, 2, 4, 5, 6)$), and even more, that it has infinitely many solutions. The amount of variables, however, makes it a complicated problem to be studied in full generality. As a result, only partial results have been obtained regarding the nature of the rational and integer solutions. It has been noted that not every positive real solution of (1) corresponds to a tetrahedron, though of course every tetrahedron corresponds to a solution. In section 1.4 we will comment on what conditions guarantee that a positive solution corresponds to an actual tetrahedron.

One of the first things that one notices about equation (1) is that if (a, b, c, d, e, f, V) is a rational solution, then so is $(ka, kb, kc, kd, ke, kf, k^3V)$ for any $k \in \mathbb{Q}$. That is, equation (1) is homogeneous in some sense. Therefore, given any rational solution one can always find an integer one by scaling it. In this sense, nothing is lost if one seeks for rational solutions instead of integer ones when the interest is to find integer solutions. More importantly, however, this implies that we only need to consider solutions up to scaling, and we will do so from now on. We will identify all the solutions that can be scaled to one another, considering them as one solution.

The approach taken to study rational tetrahedra in [Buc92] and [Chi04] is to reduce the number of variables of (1) by equating some of the edges. Buchholz [Buc92] classified partially the cases resulting from this approach and Catherine Chisholm completed this classification in her Masters thesis [Chi04]. The idea is to classify the cases resulting from equating the edges of the tetrahedron according to the number of different edge lengths the tetrahedron can have. They call "n-parameter tetrahedra" the families of tetrahedra that can have at most n different edge lengths. Classifying in this manner leads to equations with the same number of variables, though the way of equating the edges may lead to essentially different equations. This leads them to consider sub-cases for each n-parameter family.

For example, there is only one sub-case to consider regarding 1-parameter tetrahedra given by $a = b = c = d = e = f$. There are no 1-parameter rational tetrahedra as there are no rational solutions to the equation $(12V)^2 = 2a^6$, since $\sqrt{2}$ is not rational. Regarding the situation with 2-parameter tetrahedra, there are 5 sub-cases in total and Buchholz in his article [Buc92] shows that only one of these sub-cases gives rational tetrahedra. He shows there is an infinity of rational tetrahedra for this sub-case and gives a parametrization for all rational solutions of the resulting equation.

In this thesis we will study some of the sub-cases regarding 3-parameter tetrahedra. There are ten sub-cases in total. We will refer to these sub-cases as "cases" from now onwards. Catherine Chisholm's results regarding these cases are summarized in the following table.

Case	Description	No. of rational solutions
1	$a = b = c = d$	0
2	$a = c = d = f$	∞
3	$a = b = c,\ d = e$	0
4	$a = d = f,\ b = c$	0
5	$a = d = f,\ b = e$	∞
6	$a = d,\ b = e,\ c = f$	∞
7	$a = e,\ b = f,\ c = d$	0
8	$a = b,\ d = e = f$	∞
9	$a = d,\ b = f,\ c = e$	∞
10	$a = e,\ b = c,\ d = f$	0

For cases 2 and 8 there is a complete description of all rational solutions of the resulting equation given in parametric form. It is unknown whether the solutions described for case 9 exhaust all the rational solutions, although they probably do not. In Chapter 1 we will find more rational solutions for cases 5 and 6 showing that the solutions exhibited in [Chi04] do not exhaust all the solutions.

More importantly, we will take a look at the situation regarding "3-parameter" tetrahedra in a geometric context by using concepts from algebraic geometry. In particular, we will show how each of the equations resulting from equating the variables can be seen as the defining equations of projective surfaces in some weighted projective space (so in reality the term "3-parameter" should be replaced by two-dimensional).

As a result, the way to discover rational solutions will be greatly clarified, and we will show a way to use the infinite family of rational tetrahedra that are known for cases 5 and 6 to generate even more solutions of the equation. This will culminate with theorems 1.2.5 and 1.3.1 where we will show that the set of rational points is Zariski dense in the surfaces obtained from cases 5 and 6. The results will depend heavily on the fact that we are working with surfaces and that we can view these surfaces as families of elliptic curves.

The way the results from Chapter 1 were obtained depended heavily on intuition arising from the use of the concept of generic fibre of a morphism of varieties. Chapter 2 will be devoted to discussing this concept in the language of scheme theory, focusing primarily on the particular case that will be of use to analyse the situations encountered in Chapter 1. This will culminate in section 2.3 where the arithmetic problem and the geometric setting presented in Chapter 1 will be analysed once more, hopefully giving insight to the ideas that led to the results obtained in Chapter 1. Finally, in Chapter 3 we will prove some general propeties of smooth surfaces defined by equations similar to those related to "3-parameter" tetrahedra (which, however, are not smooth). In particular, we will prove that these smooth surfaces are $K3$ surfaces.

I would like to thank Professor Ronald van Luijk for his help with this thesis. It would have been impossible for me to create it without his guidance, support, and overwhelming enthusiasm. Thank you Ronald!

Chapter I

RATIONAL TETRAHEDRA

1.1 The Variety Defined by the Equation

Following [Chi04], and as explained in the introduction, we will describe a tetrahedron by specifying the lengths of its edges a, b, c, d, e, f which are to be arranged in the manner depicted in Figure 1.

The focus of this thesis will be on tetrahedra with rational sides and volume. Accordingly, we define the following.

1.1.1 Definition. A *rational tetrahedron* is a tetrahedron with rational sides and volume.

In section 1.4 we will comment why not every positive real solution of the equation relating the volume and edges of a tetrahedron given in (1) corresponds to a tetrahedron. For now we will focus on finding rational solutions to the equation without worrying whether or not a solution corresponds to a tetrahedron.

As was mentioned in the introduction, we identify any two solutions of (1) that can be scaled to one another. In the language of algebraic geometry one can accomplish this identification by considering (1) as the defining equation of an algebraic variety X over a field K in weighted projective[1] space $\mathbb{P}_K(1,1,1,1,1,1,3)$, defined as the set $K^7 - \{(0,0,0,0,0,0,0)\}$ modulo the equivalence relation $(a,b,c,d,e,f,V) \sim (ka,kb,kc,kd,ke,kf,k^3V)$ for any $k \in K^*$. We will denote this space simply as $\mathbb{P}(1,1,1,1,1,1,3)$ when the field of definition is clear. The elements of this weighted projective space will be written as $[a:b:c:d:e:f:V]$.

---

[1] In general, $\mathbb{P}_K(d_0,\ldots,d_n)$ with coordinates $x_0,\ldots,x_n$ is the weighted projective n-space defined as $K^{n+1} - \{(0,\ldots,0)\}$ modulo the equivalence relation

$$(x_0,\ldots,x_n) \sim (k^{d_0}x_0,\ldots,k^{d_n}x_n)$$

for any $k \in K^*$.

In this language, finding the rational solutions to (1) then amounts to finding rational points on the variety X. Even more, if the interest is to find points corresponding to "non-degenerate" tetrahedra (that is, with none of the sides equal to 0), then studying any of the standard affine charts of X (that is, given by some coordinate not equal to zero) will suffice.

1.2 Case 5 *of Buchholz's Classification*

As mentioned in the introduction, these are tetrahedra (a, b, c, d, e, f, V) that satisfy the extra conditions $a = d = f$, $b = e$. Substituting these into (1) gives

$$(12V)^2 = -(a^2 + 2b^2 - c^2)(a^2 - b^2 - ac)(a^2 - b^2 + ac),$$

which after introducing $y = 12V$ gives the equation of a weighted projective surface S defined by

$$S : \quad y^2 = -(a^2 + 2b^2 - c^2)(a^2 - b^2 - ac)(a^2 - b^2 + ac) \tag{2}$$

in weighted projective space $\mathbb{P}(1, 1, 1, 3)$ with coordinates a, b, c, y.

1.2.1 An Affine Part of the Surface

We will take a closer look at the affine part of S of points satisfying $a \neq 0$ which we will denote by S_a and which is defined by

$$S_a : \quad y_1^2 = (1 + 2\lambda^2 - x^2)(x - (1 - \lambda^2))(x + (1 - \lambda^2))$$

where $\lambda = b/a$, $y_1 = y/a^3$ and $x = c/a$ are affine coordinates[2] in $\mathbb{A}^3$.

1.2.1 Theorem. *The surface S_a is birationally equivalent to the surface*

$$E : \quad v^2 = u^3 + A(\lambda)u + B(\lambda) \tag{3}$$

with coordinates u, v, λ (the λ-coordinate is the same in both S_a and E), where

$$\begin{aligned} A(\lambda) &= -(16 - 32\lambda^4 + 24\lambda^6 + \lambda^8)/3 \\ B(\lambda) &= 2(2 + \lambda^4)(-32 + 112\lambda^4 - 72\lambda^6 + \lambda^8)/27. \end{aligned}$$

[2] It will become clear why we have chosen these awkward names for the coordinates in what follows.

Proof. Note that the defining equation of S_a also defines a curve over $\mathbb{Q}(\lambda)$ which is of the form $y_1^2 = f(x)$ with $f(x)$ a polynomial of degree 4 in x with no repeated roots over $\overline{\mathbb{Q}(\lambda)}$ and a point $O = (x, y_1) = (\lambda^2 - 1, 0)$ defined over $\mathbb{Q}(\lambda)$. It is known that over any field k, an equation of the form $y^2 = f(x)$ with $f(x) \in k[x]$ a polynomial of degree 4 in x with a root in k and no repeated roots over $\overline{k}$ is an elliptic curve (see Appendix A), and there are standard ways to bring this equation into Weierstrass form by taking one of the points corresponding to a root of $f(x)$ in k to infinity. We will do this for the equation of S_a regarding it as that of the elliptic curve over $\mathbb{Q}(\lambda)$ to illustrate how the process is done[3] .

With this at mind, we will use subindices on the variables and on the polynomial f to keep track of each transformation to be made. We begin with the equation $y_1^2 = f(x)$ for which we will rename x as x_1 and f as f_1.

First homogenize the equation $y_1^2 = f_1(x_1)$ to obtain

$$Y_1^2 Z_1^2 = Z_1^4 f_1(X_1/Z_1)$$

in $\mathbb{P}^2(\mathbb{Q}(\lambda))$ where $O = [X_1 : Y_1 : Z_1] = [\lambda^2 - 1 : 0 : 1]$ and then make the change of variables

$$\begin{aligned} X_1 &= (\lambda^2 - 1)X_2 \\ Y_1 &= Y_2 \\ Z_1 &= X_2 + Z_2 \end{aligned}$$

sending O to $[1 : 0 : 0]$. The curve with new projective coordinates X_2, Y_2, Z_2 has affine part $Z_2 \neq 0$ (i.e., $y_2 = Y_2/Z_2$, and $x_2 = X_2/Z_2$) given by

$$y_2^2(x_2 + 1)^2 = f_2(x_2)$$

where f_2 is a polynomial of degree 3 in $\mathbb{Q}(\lambda)[x_2]$ whose leading coefficient is $c = 2\lambda^2(\lambda^2 - 1)(\lambda^2 - 4)$. To remove this coeficient and the $(x_2 + 1)^2$ term on the left, multiply the whole equation by c^2 and rewrite the equation in terms of the new variables $x_3 = cx_2$ and $y_3 = c(x_2 + 1)y_2$. This gives an equation of the form

$$y_3^2 = f_3(x_3)$$

[3]The transformation shown in Appendix A could have also been used, but we prefered to use projective coordinates explicitly here.

where f_3 is a monic cubic polynomial in $\mathbb{Q}(\lambda)[x_3]$. To remove the x_3^2 term in f_3, take its coeficient $d = \lambda^4 - 12\lambda^2 - 4$ and make the change of variable $x_4 = x_3 + d/3$. The resulting equation

$$y_3^2 = f_4(x_4)$$

is the one stated in the theorem after renaming the variables $u = x_4$, $v = y_3$.

The total transformation between these curves is given by

$$\begin{aligned} v &= \frac{2y_1\lambda^2(\lambda^2-1)(\lambda^2-4)}{(\lambda^2-1-x)} \\ u &= \frac{\lambda^6+5\lambda^4x-13\lambda^4-12\lambda^2x+8\lambda^2+4x+4}{3(\lambda^2-1-x)} \end{aligned} \tag{4}$$

with inverse

$$\begin{aligned} x &= \frac{\lambda^2-1}{4-12\lambda^2+5\lambda^4+3u} \\ y_1 &= \frac{18v\lambda^2(\lambda^2-4)(\lambda^2-1)}{(4-12\lambda^2+5\lambda^4+3u)^2}. \end{aligned} \tag{5}$$

Returning to the surfaces, these equations show that the surface E in $\mathbb{A}^3$ with coordinates u, v, λ defined by equation (3) is birationally equivalent to S_a. The rational maps between them are given by (4) and (5) and the identity on λ. ■

Note that if we consider the morphism

$$\begin{aligned} \sigma : \quad S_a \quad &\to \mathbb{A}^1 \\ (x, \lambda, y_1) \quad &\mapsto \lambda \end{aligned}$$

then the previous theorem implies that the *fibre* above $\lambda \in \mathbb{A}^1$ (that is, $\sigma^{-1}(\lambda)$) is an elliptic curve over $\mathbb{Q}$ for all but finitely many $\lambda \in \mathbb{A}^1(\mathbb{Q})$. This fact will be exploited in the following sections and the situation will be dealt with in Chapter 2 and analyzed in more detail using more heavy machinery. A very schematic diagram of the situation looks something like that depicted in Figure 2.

The fibres that are not elliptic curves occur above the values of λ where the discriminant $\Delta = 4A^3(\lambda) + 27B^2(\lambda)$ vanishes. The discriminant is computed to be

$$\Delta(\lambda) = \lambda^8(\lambda-2)^4(\lambda+2)^4(\lambda-1)^2(\lambda+1)^2(2\lambda^2+1).$$

If $\Delta(\lambda_0) = 0$ then the fibre above λ_0 will be a union of rational curves defined over $\overline{\mathbb{Q}}$. In this thesis we will refer to rational curves as *parametrizable* curves due to the fact that

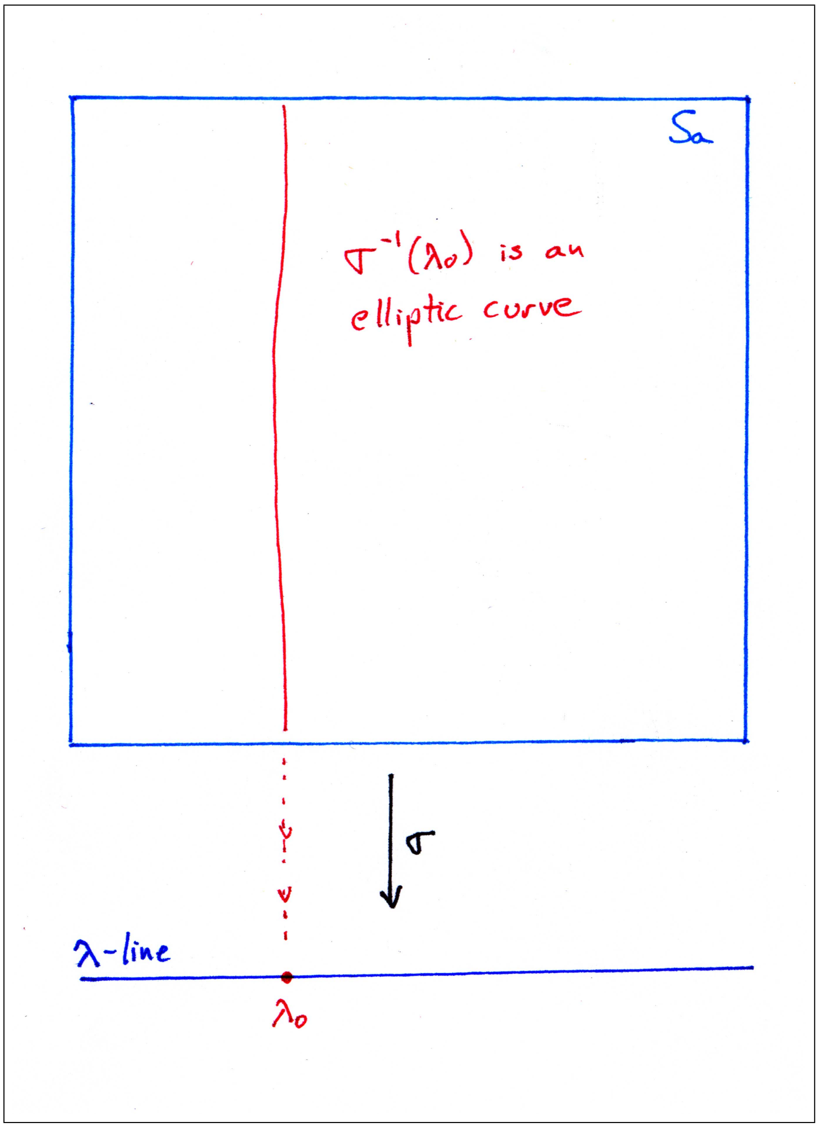

Figure 2: σ and its fibres.

the term rational already has an important meaning. Some of these fibres could contain infinitely many rational points, but we will be interested here in parametrizable curves with points in infinitely many fibres. In the following section we will give parametrizations of two such curves, each with an infinite number of rational points.

1.2.2 Parametrizable Curves on the Surface

By setting $a = c$ in (2) one obtains the equation $y^2 = 2b^4(2a^2 - b^2)$ which defines a curve that is birationally equivalent to a conic with a trivial rational point. Specifically, setting $r = y/ab^2$ and $s = b/a$ we obtain

$$4 = r^2 + 2s^2$$

with the point $(s, r) = (0, 2)$. Using the chord method one finds a parametrization for this conic in terms of a parameter t, namely

$$s(t) = \frac{4t}{t^2 + 2}, \qquad r(t) = \frac{4 - 2t^2}{t^2 + 2}.$$

Setting momentarily $a(t) = c(t) = 1$ and finding $b(t)$, $y(t)$ from the equations $r = y/ab^2$ and $s = b/a$ one finds a parametrization for C_1 which can be scaled using the homogeneity of the equation of S by multiplying by $t^2 + 2$ (and y by $(t^2 + 2)^3$) giving

$$\begin{aligned} a(t) &= t^2 + 2 \\ b(t) &= 4t \\ c(t) &= a(t) \\ y(t) &= 32t^2(2 - t^2) \end{aligned}$$

Setting $c = 3b/2$ gives another parametrizable curve C_2 on S with a rational point[4] . The parametrization of C_2 is given by

$$\begin{aligned} a(t) &= 4(t^2 - 1) \\ b(t) &= 2(t^2 + 1) \\ c(t) &= 3(t^2 + 1) \\ y(t) &= 8t(3t^2 - 5)(5t^2 - 3). \end{aligned}$$

[4]This curve comes from [Chi04], and was probably found using a fibration of the sort shown earlier but setting $c = \lambda b$ and analyzing the singular fibres.

Thus, we already have an infinite number of rational points on S lying on C_1 and C_2 since every $t \in \mathbb{Q}$ gives a rational point on each of these curves. In the next section however, we will show that both curves generate an infinite family of rational curves on S, each containing and infinite number of rational points, thus proving that the set of rational points on S is Zariski dense.

1.2.3 The Set of Rational Points is Zariski Dense

Every parametrizable curve T lying inside S, given say as $[a(t) : b(t) : c(t) : y(t)]$ has an affine part lying on S_a which we will also denote by T given by

$$(x(t), \lambda(t), y_1(t)) = \left(\frac{c(t)}{a(t)}, \frac{b(t)}{a(t)}, \frac{y(t)}{a^3(t)}\right).$$

If we substitute $\lambda(t)$ for λ in the equation for the surface E from theorem 1.2.1 we obtain an equation of the form $v^2 = u^3 + A_T(t)u + B_T(t)$ where $A_T(t) = A(\lambda(t))$ and $B_T(t) = B(\lambda(t))$. If the discriminant $4A_T^3(t) + 27B_T^2(t)$ does not vanish, this equation will define an elliptic curve E_T over the field $k(t)$ where $a(t), b(t), c(t), y(t) \in k(t)$, given by

$$E_T : \quad v^2 = u^3 + A_T(t)u + B_T(t).$$

Transformation (4) then gives a point $P_T = (u_T(t), v_T(t))$ on E_T corresponding to T defined over $k(t)$. Moreover, every point on $E_T(k(t))$ gives rise to a parametrizable curve on S_a over k. Specifically, if $(u(t), v(t))$ is a point on $E_T(k(t))$, then by (5) we obtain a curve on S_a parametrized by

$$\begin{aligned} \lambda(t) &= \lambda(t) \\ x(t) &= \frac{\lambda^2(t) - 1}{4 - 12\lambda^2(t) + 5\lambda^4(t) + 3u(t)} \\ y_1(t) &= \frac{18v\lambda^2(t)(\lambda^2(t) - 4)(\lambda^2(t) - 1)}{(4 - 12\lambda^2(t) + 5\lambda^4(t) + 3u(t))^2}. \end{aligned} \tag{6}$$

Therefore, using the group structure on E_T we can find "multiples" of T in S_a by finding multiples of P_T in E_T and then using (6). If T is parametrized over[5] $\mathbb{Q}$ (note that this implies that T has infinitely many rational points), then every multiple of T will also be parametrized over $\mathbb{Q}$ as both the group structure on E_T and the map (6) will be defined by rational functions over $\mathbb{Q}(t)$. The set Γ of parametrizable curves constructed by

[5] That is, $a(t), b(t), c(t), y(t) \in \mathbb{Q}(t)$.

finding multiples of T in S_a in this way could be $\{T\}$, or there could be some non-trivial multiples of T. It is simply a matter of checking.

As an illustration we will prove that $2 \cdot C_1 \neq C_1$. For the curve C_1 we have $\lambda(t) = 4t/(t^2+2)$. The curve C_1 corresponds to the point

$$\begin{aligned} u_{C_1}(t) &= -\frac{4(t^{12}+4t^{10}-260t^8-544t^6-1040t^4+64t^2+64)}{3(t^2-2)^2(t^2+2)^4} \\ v_{C_1}(t) &= -1024\frac{(t^2-4t+2)(t^2-2t+2)(t^2+2t+2)(t^2+4t+2)t^4}{(t^2-2)^3(t^2+2)^5}. \end{aligned}$$

on the elliptic curve

$$E_{C_1}: \quad v^2 = u^3 + A_{C_1}(t)u + B_{C_1}(t) \tag{7}$$

where

$$\begin{aligned} A_{C_1}(t) &= -\frac{16}{3}\frac{t^{16}+16t^{14}-400t^{12}+2496t^{10}+17504t^8+9984t^6-6400t^4+1024t^2+256}{(t^2+2)^8} \\ B_{C_1}(t) &= -\frac{128}{27}\frac{(t^8+8t^6+152t^4+32t^2+16)(t^{16}+16t^{14}-784t^{12}+2496t^{10}+14432t^8+9984t^6-12544t^4+1024t^2+256)}{(t^2+2)^{12}}. \end{aligned}$$

If we duplicate this point and then transform it back to S_a and subsequently to S, we find the multiple $2 \cdot C_1$ of the parametrizable curve. The result of this computation is

$$\begin{aligned} a(t) &= t^2+2 \\ b(t) &= 4t \\ c(t) &= \frac{(t^2-4t+2)(t^2+4t+2)(t^8+20t^6-56t^4+80t^2+16)}{(t^2+2)(t^8-12t^6+72t^4-48t^2+16)} \\ y(t) &= -2^8 3\frac{t^4(t^2-4t+2)(t^2-2t+2)(t^2-6)(t^2-2)(t^2-\frac{2}{3})(t^2+2t+2)(t^2+4t+2)}{(t^8-12t^6+72t^4-48t^2+16)^2}. \end{aligned}$$

Therefore, $2 \cdot C_1 \neq C_1$ and we have found more rational points on S.

It can even happen that with this group structure the curve T has infinite order giving rise to an infinite number of parametrizable curves lying inside S, each with an infinite number of rational points when T can be parametrized over $\mathbb{Q}$. The following theorem then comes at hand.

1.2.2 Theorem (Nagel-Lutz)**.** *Let $E/\mathbb{Q}(t)$ be an elliptic curve given by*

$$E: \quad y^2 = x^3 + a_2x^2 + a_4x + a_6$$

where $a_2, a_4, a_6 \in \mathbb{Q}[t]$. If $P = (x_P, y_P) \in E(\mathbb{Q}(t))$ is a torsion point then $x_P, y_P \in \mathbb{Q}[t]$.

Proof. There seems to be no reference in the literature for this theorem in the form stated. We refer the reader to [vL00] p. 58, for a sketch of the proof, or to [ST92] where the proof of the theorem can be followed by replacing $\mathbb{Q}$ by $\mathbb{Q}(t)$ and $\mathbb{Z}$ by $\mathbb{Q}[t]$ in the proof of the classical theorem. ■

With the aid of this theorem we will prove the following.

1.2.3 Corollary. *The points $P_{C_1} \in E_{C_1}(\mathbb{Q}(t))$ and $P_{C_2} \in E_{C_2}(\mathbb{Q}(t))$ corresponding to the curves C_1 and C_2 have infinite order.*

Proof. The main idea is to show that for $T = C_i$, $i = 1, 2$, one can transform the equation of E_T so that $A_T(t)$ and $B_T(t)$ become polynomials by an appropriate change of coordinates removing the denominators of A_T and B_T. It is then easily checked that after this transformation the point P_T corresponding to T does not have coordinates in $\mathbb{Q}[t]$ in either of the cases. By Theorem 1.2.2 it will then follow that the curves have infinite order.

We will prove the result for C_1 as the computations for C_2 are similar. The change of variables making $A_{C_1}(t), B_{C_1}(t)$ belong to $\mathbb{Q}[t]$ is obtained by multiplying the whole equation by $(t^2+2)^{12}$ and setting $u' = (t^2+2)^4 u$ and $v' = (t^2+2)^6 v$, which gives the elliptic curve

$$E'_{C_1}: \quad v'^2 = u'^3 + A'_{C_1}(t)u' + B'_{C_1}(t)$$

where

$$\begin{aligned} A'_{C_1}(t) &= A_{C_1}(t)(t^2+2)^8 \in \mathbb{Q}[t] \\ B'_{C_1}(t) &= B_{C_1}(t)(t^2+2)^{12} \in \mathbb{Q}[t] \end{aligned}$$

and where C_1 corresponds to the point $P'_{C_1} = (u'_{C_1}(t), v'_{C_1}(t))$ given by

$$\begin{aligned} u'_{C_1}(t) &= -\frac{4(t^{12}+4t^{10}-260t^8-544t^6-1040t^4+64t^2+64)}{3(t^2-2)^2} \\ v'_{C_1}(t) &= -1024\frac{(t^2+2)(t^2-4t+2)(t^2-2t+2)(t^2+2t+2)(t^2+4t+2)t^4}{(t^2-2)^3}, \end{aligned}$$

which does not have coordinates in $\mathbb{Q}[t]$. ■

The existence of such an infinite set of curves, each with infinitely many rational points implies that the set of rational points on S is Zariski dense as the following proposition shows.

1.2.4 Proposition. *Let X be an irreducible surface. If Γ is an infinite set of irreducible curves contained in X, then $\bigcup \Gamma$ is a Zariski dense subset of X.*

Proof. The closure Ω of $\bigcup \Gamma$ contains all the curves, which by hypothesis are irreducible. If $\dim \Omega = 1$ this would imply that the maximal irreducible closed subsets of Ω are curves, but Ω is closed and therefore can be expressed as a finite union of its maximal irreducible closed subsets (see [Har77], Proposition I.1.5). However, no finite union of the curves is equal to Ω as any curve intersects any other in only a finite number of points. Therefore, $\dim \Omega = 2$ and since X is irreducible this implies that $\Omega = X$. ■

With this at hand we can prove the main result about rational points on S.

1.2.5 Theorem. *The set of rational points on S is Zariski dense.*

Proof. The set $\Gamma = \{n \cdot C_1 : n \in \mathbb{N}\}$ is an infinite set of curves contained in S, and Proposition 1.2.4 implies that

$$\bigcup \Gamma = \bigcup_{n=1}^{\infty} n \cdot C_1$$

is a Zariski dense subset of S. The set of rational points in $n \cdot C_1$ is Zariski dense in $n \cdot C_1$ since every $n \cdot C_1$ has an infinity of rational points and any infinite set of points in a curve is Zariski dense. From this it follows that the set of rational points, which is a dense subset of $\bigcup \Gamma$, is dense in S. ■

1.3 Case 6 *of Buchholz's Classification*

A much more symmetric equation for another projective surface related to rational tetrahedra is obtained when equating the opposite edges of the tetrahedron (i.e., $a = d$, $b = e$, $c = f$). The resulting equation (after renaming $y = 12V$) is

$$y^2 = 2(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2).$$

Here again, we can define a fibration of the surface where almost all the fibres are elliptic curves, this time with $\lambda = c/(a-b)$. Specifically, by setting $\lambda = c/(a-b)$, $x = a/b$ and $y_1 = y/b^2(a-b)$ we arrive at the equation

$$y_1^2 = 2[x^2(1-\lambda^2) + 2\lambda^2 x + 1 - \lambda^2][x(1+\lambda^2) - (\lambda^2 - 1)][x(\lambda^2 - 1) - (\lambda^2 + 1)]$$

which again is of the form $y_1^2 = f(x)$ with $f(x) \in \mathbb{Q}(\lambda)[x]$ of degree 4 and no repeated roots over $\overline{\mathbb{Q}(\lambda)}$. The curve defined by the equation over the field $\mathbb{Q}(\lambda)$ contains the point

$O = (x, y_1) = ((\lambda^2-1)/(\lambda^2+1), 0)$ and so is an elliptic curve over $\mathbb{Q}(\lambda)$. Since the discriminant of this curve is a non constant polynomial over $\mathbb{Q}$, all but finitely many fibres of the morphism projecting the surface on the λ-coordinate are elliptic curves over $\mathbb{Q}$.

Sending the point O to infinity, this time using the transformation found in [ST92] with $\beta = 1$ and making further transformations to make the leading coeficient equal to 1, we finally end up with the equation[6]

$$v^2 = u^3 + a_1(\lambda)u^2 + a_2(\lambda)u + a_3(\lambda) \tag{8}$$

where

$$\begin{aligned} a_1(\lambda) &= 2^2(\lambda^2+1)(\lambda^6 - 15\lambda^4 + 7\lambda^2 - 1) \\ a_2(\lambda) &= -2^2(\lambda^2-1)(5\lambda^2-1)(\lambda^2+1)^4 \\ a_3(\lambda) &= -2^9(\lambda^2-1)^2(\lambda^2+1)^7. \end{aligned}$$

The parametrizable curve

$$\begin{aligned} a(t) &= t^2+2 \\ b(t) &= 4t \\ c(t) &= t^2+2 \\ y(t) &= 32t^2(2-t^2) \end{aligned}$$

on the surface is found by setting $a = c$ in a way similar to what was done with C_1 and C_2 (In fact, the curve is C_1, but it is now embedded into another surface).

This curve corresponds to a point on the elliptic curve over $\mathbb{Q}(t)$ given by

$$v^2 = u^3 + b_1(t)u^2 + b_2(t)u + b_3(t)$$

obtained from (8) by replacing λ by $\lambda(t) = c(t)/(a(t) - b(t))$ (i.e., $b_i(t) = a_i(\lambda(t))$).

The denominators of $b_1(t)$, $b_2(t)$ and $b_3(t)$ are $(t^2-4t+2)^8$, $(t^2-4t+2)^{12}$ and $(t^2-4t+2)^{18}$ respectively, and the coordinates of the point corresponding to the curve are given by

$$\begin{aligned} v(t) &= \frac{2^{16}t^4(2+t^2)^2(2-2t+t^2)^2(4-8t+12t^2-4t^3+t^4)^3}{(-2+t^2)^3(2-4t+t^2)^{11}} \\ u(t) &= \frac{2^{12}t^3(2+t^2)^2(2-2t+t^2)^2(4-8t+12t^2-4t^3+t^4)}{(-2+t^2)^2(2-4t+t^2)^7}. \end{aligned}$$

[6] See Appendix B for the explicit computations.

Then, the change of variables to make the $b_i(t)$ polynomials in t is obtained by multiplying the equation by a suitable power of $t^2 - 4t + 2$ under which the coordinates of the point on the elliptic curve over $\mathbb{Q}(t)$ corresponding to the curve will still have polynomial denominators. An application of Proposition 1.2.2 then proves that the point corresponding to the curve has infinite order, and therefore gives rise to an infinite set of parametrizable curves on the surface, each with infinitely many rational points. This gives the following result by Proposition 1.2.4.

1.3.1 Theorem. *The set of rational points on the surface related to tetrahedra of case* 6 *is Zariski dense.*

1.4 Rational Solutions Corresponding to Realizable Tetrahedra

As mentioned in section 1.1, not every rational solution to (1) corresponds to a rational tetrahedron, not even a positive solution. The reason why this is so is that with the values (a, b, c, d, e, f) of a solution it may not be possible to form the triangles which will be the faces of the tetrahedron. The general tetrahedron has triangle faces abc, aef, bdf and cde, and a positive rational solution to (1) may give values with which some of the faces do not exist. For example no triangle abc exists for the values $(a, b, c) = (2, 9, 5)$, and this is due to the fact that the triangle inequality is not satisfied for one of the sides $(9 \not< 2 + 5)$. The condition necessary for a triangle with edges a, b, c to exist is that all three triangle inequalities $a < b + c$, $b < a + c$ and $c < a + b$ hold.

In another direction (but one that concerns us less as will be explained below), there exist triangular faces abc, aef, bdf and cde for which no tetrahedron exists. The geometric reason behind this fact is that it can happen that the face Δ with the largest area has area greater than the sum of the remaining three, so when constructing the tetrahedron with Δ lying down on a table and laying the other triangles on their respective bases, their edges will not meet above Δ and they will simply collapse on top of Δ. In [DS92] it is proved that if the edges (a, b, c, d, e, f) define realizable triangular faces for the tetrahedron (that is, all triangle inequalities for abc, aef, bdf and cde are satisfied), then they will define a realizable tetrahedron if and only if the right hand side of equation (1) relating the volume of the tetrahedron to its edges is positive. An example of a solution where this fails is given by $a = b = c = 14$ and $d = e = f = 8$ where the right hand side gives $-153,664$.

As our focus here was on finding rational solutions for the whole equation which includes the right hand side being the square of a rational number, the previous condition is always satisfied. Therefore, the rational solutions we have found will be realizable tetrahedra if and only if both the edges and V are positive and the triangle inequalities for all the faces are satisfied.

As an illustration consider case 5. The tetrahedra from case 5 only have two types of faces given by the triangles aab and abc. For the face aab to exist we need $b < 2a$ as the other inequalities are trivially satisfied if the edges are positive. For the face abc to exist we need $a < b + c$, $b < a + c$ and $c < a + b$.

We will call a point $[a : b : c : y]$ on the projective surface S related to Case 5 a *positive* point if it has a representative where all the coordinates are positive (remember that $y = 12V$). Note that this is equivalent to requiring that all the ratios of its coordinates are positive, or that its affine coordinates x, λ, y_1 are positive if $a \neq 0$.

We can now take a look at the parametrizable curve C_1 to see if it gives any tetrahedra. The curve C_1 was defined by $a = c$ on the surface and was given by

$$\begin{aligned} a(t) &= t^2 + 2 \\ b(t) &= 4t \\ c(t) &= a(t) \\ y(t) &= 32t^2(2 - t^2). \end{aligned}$$

For a point on C_1 to be positive we need $0 < t < \sqrt{2}$ since $a(t), c(t)$ are always positive and so $b(t)$ and $y(t)$ have to be positive. It is easily checked that both $b < 2a$ and $b < a + c$ are always satisfied (in fact, they are the same inequality), and that $a < b + c$ and $c < a + b$ are satisfied only when $t > 0$. Therefore, only the rational points on C_1 given by $0 < t < \sqrt{2}$ give rational tetrahedra.

It may be the moment to note the subtleties regarding the use of weighted projective coordinates on our surface. The curve C_1 on S was defined by $a = c$ and so all but finitely many points on the surface S satisfying $a = c$ will be obtained from the parametrisation we found. Substituting $t = 3$ gives the values $(a, b, c, y) = (11, 12, 11, -2016)$ which do not define a tetrahedron only because of the fact that $y < 0$. There then must be a point P on C_1 corresponding to $(a, b, c, y) = (11, 12, 11, 2016)$ (unless we are really unlucky) giving this rational tetrahedron as it satisfies $a = c$, but it is not obvious what the value of t is at a first glance.

The reason for this is that we are working with projective coordinates, so $P = [11 : 12 : 11 : 2016]$ will be given by the parametrization of C_1 as $P = [11r : 12r : 11r : 2016r^3]$ for some $r \in \mathbb{Q}$. With a little algebra one finds $t = 2/3$ and $r = 2/9$ giving the rational tetrahedron with edges $(a, b, c) = (22/9, 24/9, 22/9) = 2/9(11, 12, 11)$. Of course $t = 2/3$ lies in the range $0 < t < \sqrt{2}$ where we said all tetrahedra where obtained.

The situation however is more complicated if one takes a look at the new curve $2 \cdot C_1$ that we obtained by duplicating the curve C_1, let alone the infinite set of curves generated by C_1. Our computations showed that $2 \cdot C_1$ was parametrized as

$$\begin{aligned}
a(t) &= t^2 + 2 \\
b(t) &= 4t \\
c(t) &= \frac{(t^2 - 4t + 2)(t^2 + 4t + 2)(t^8 + 20t^6 - 56t^4 + 80t^2 + 16)}{(t^2 + 2)(t^8 - 12t^6 + 72t^4 - 48t^2 + 16)} \\
y(t) &= -2^8 3 \frac{t^4(t^2 - 4t + 2)(t^2 - 2t + 2)(t^2 - 6)(t^2 - 2)(t^2 - \frac{2}{3})(t^2 + 2t + 2)(t^2 + 4t + 2)}{(t^8 - 12t^6 + 72t^4 - 48t^2 + 16)^2}
\end{aligned}$$

and then the verifications of the triangle inequalities for the face areas becomes a complicated problem. We will not deal here with the issue of when points of $2 \cdot C_1$ nor any multiple of C_1 are rational tetrahedra. Some analysis on what the group operation does to the curve on the surface is necessary as otherwise each multiple of C_1 is a situation that must be analyzed by itself.

A relevant fact for anyone wishing to pursue this problem is that all the triangle inequalities for a triple (a, b, c) are satisfied if and only if the right hand side of equation (9) given in section 1.6 (below) is positive.

1.5 The General Case

The previous sections show some partial results on the existence and nature of rational tetrahedra under certain restrictions. As was mentioned in the introduction, in [Chi04] all the possible surfaces obtained by equating the edges are studied and the question of whether there exist infinitely many, finitely many or no rational tetrahedra under each condition is answered. In the previous sections we extended the results in two of the cases using the known points to generate others, we set the whole situation in a much more geometric setting[7] , and we proved that the set of rational solutions is dense in the Zariski topology of the surfaces.

[7] There is in fact no mention of the surfaces defined by the equations in [Chi04], nor in any reference in the relevant literature we know of.

It would be desirable of course to study the equation in full generality but the amount of variables makes the problem difficult. It is clear that the results obtained depended heavily on reducing the number of variables, in this way making it possible to use the theory of rational and elliptic curves and ultimately that of elliptic surfaces.

1.6 Heron Tetrahedra

A natural generalization of the problem regarding tetrahedra with rational edges and volume is to consider the problem of when the edges, volume and *face areas* are rational. These are known as *heron tetrahedra* (though the term is still not universal) as a natural extension of the term *heron triangle* for a triangle with rational sides and area in honor of the equation relating the edges of a triangle and its area from Heron of Alexandria (60 A.D.). This problem will not be undertaken in this thesis. Nonetheless, we will give the relevant equations and will also show a very interesting surface related to a simplification of this problem.

We begin by stating the relevant results regarding heron triangles.

1.6.1 Heron Triangles

The formula relating the sides a, b, c of a triangle with its area A is given by

$$(4A)^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \tag{9}$$

and is known as *heron's formula*. Triangles with rational edges and area are known as *heron triangles* and have been studied for several centuries.

In a way similar to what was done with the equations studied previously for tetrahedra, equation (9) can be viewed as the defining equation of a surface H in weighted projective space $\mathbb{P}(1,1,1,2)$ with coordinates $[a : b : c : y]$ where $y = 4A$. Positive rational points on H (in the projective sense) satisfying the triangle inequalities correspond to heron triangles modulo scaling.

A very well known fact about this surface is that it is parametrizable over $\mathbb{Q}$. Therefore, all heron triangles are known and we have a parametrization of them. The first parametrization was given by Euler[8] (see, [Dic52] p. 193), but we will be concerned here with the following one from [Car59],

[8]Of course, Euler did not state this fact geometrically as we have just done. Nonetheless, it is neither stated in [Car59] in this way nor in most books on number theory. Our use of algebraic geometry is extensive in this thesis in order to illustrate the usefulness it has for number theory.

$$
\begin{aligned}
a &= n(m^2+k^2) \\
b &= m(m^2+k^2) \\
c &= (m+n)(mn-k^2) \\
y &= knm(m+n)(mn-k^2)
\end{aligned}
\tag{10}
$$

which gives all the heron trianlges (i.e., the edges are positive and the triangle inequalities are satisfied) for $k, m, n > 0$ and $k^2 > mn$.

1.6.2 An Interesting Surface

The parametrization of heron triangles (10) naturally suggests itself as a starting point for searching heron tetrahedra, specifically for case 6 as the faces of these tetrahedra are all congruent, and so there is only one more equation to deal with.

If one wishes to search for heron tetrahedra in this family, one might as well start off by making sure the surface areas are rational by substituting this parametrization, as then one only needs to make sure the volume is rational. The resulting equation

$$(12V)^2 = (4mn(m+n)(mn-k^2))^2(k^2(m+n)^2-(mn-k^2)^2)(m^2-k^2)(n^2-k^2)$$

defines a surface that is birationally equivalent to the projective surface with coordinates $[k : m : n : y]$ defined by

$$y^2 = (k^2(m+n)^2-(mn-k^2)^2)(m^2-k^2)(n^2-k^2)$$

by setting $y = 12V/(4mn(m+n)(mn-k^2))$.

All heron tetrahedra with equal opposite edges come from a rational point on this surface as the parametrization gives all heron triangles (modulo scaling) and therefore all rational face areas a tetrahedron can have. This surface has been partly studied in [Buc92] and [Chi04] but there is still a lot to be done.

We cite here the equations showing how to define a fibration for this surface where almost all fibres are elliptic curves. In this way, this seemingly more complicated problem has been reduced into one in a familiar setting.

The affine part of the surface corresponding to $k \neq 0$ is given by

$$z^2 = ((x+\lambda)^2-(x\lambda-1)^2)(x^2-1)(\lambda^2-1)$$

where $z = y/k^2$, $x = m/k$, $\lambda = n/k$ which is again seen to define an elliptic curve over $\mathbb{Q}(\lambda)$. It can be seen that a change of coordinates sending the point $(x, z) = (1, 0)$ to infinity yields the following equation in Weierstrass form.

$$v^2 = u(u + \lambda^2 - 1)(u - \lambda^2(\lambda^2 - 1)) \tag{11}$$

In this manner, any known heron tetrahedron from Case 6 gives a value of λ and therefore a point on a curve E_λ defined over $\mathbb{Q}$ defined by (11). E_λ is elliptic for all but finitely many values of λ. It would be desirable to know when the point has infinite order and when the points of the generated subgroup correspond to heron tetrahedra but the problem is yet to be settled.

Chapter II

FIBRATIONS AND GENERIC FIBRES

The main purpose of this chapter is to study the situation encountered in Chapter 1 in an even more geometric setting. We will take a closer look at the geometry behind fibrations of surfaces, focusing mainly on explaining the concept of *generic fibre*. With the aid of this concept we will give some insight on the geometry behind the results obtained in Chapter 1.

Generic fibres will be discussed in the modern language of schemes, where their definition has a strong geometric meaning. The reader is assumed to have a basic knowledge of scheme theory which can be found on the first sections of Chapter 2 of Hartshorne's book [Har77], though he/she may skip the technical details regarding schemes and focus on understanding the concept of the generic fibre of a morphism projecting an affine hypersurface onto one of its coordinates.

2.1 Varieties and Schemes

Every affine variety V has a corresponding affine scheme given by Spec $A(V)$ where $A(V)$ is the coordinate ring of V. Even though modern algebraic geometers identify these two objects and refer to one or the other interchangeably, in this thesis we will refer to Spec $A(V)$ as the scheme *associated* to the variety V for the sake of clarity. We will continue to call varieties the objects from classical algebraic geometry and whenever we want to refer to the associated scheme we will do so explicitly. For any two affine varieties V and W and a morphism $\sigma : V \to W$ between them, there is an induced ring homomorphism between their coordinate rings $\hat{\sigma} : A(W) \to A(V)$ given by $f \mapsto f \circ \sigma$. With $\hat{\sigma}$ we can define a map between the associated schemes $\bar{\sigma} : \text{Spec } A(V) \to \text{Spec } A(W)$ by sending every prime ideal of $A(V)$ to its inverse image under $\hat{\sigma}$. This induced map is in fact a morphism of schemes.

The main construction of schemes other than affine ones is the construction of Proj S for a graded ring S. In the case where $S = k[x_0, \ldots, x_n]$ with the usual grading (k of degree 0 and $x_1, \ldots, x_n$ of degree 1) the scheme Proj S is the scheme associated to the usual projective n-space $\mathbb{P}^n_k$. However, the construction of Proj S in its full generality allows one to construct schemes other than the one associated to regular projective space. For example, if we define on $S = k[x_0, \ldots, x_n]$ a different grading by setting the degree of the elements of k equal to 0 and giving the indeterminate x_i degree d_i, then Proj S is a weighted projective space such as those encountered in Chapter 1.

As a matter of fact, we can give each variety encountered in Chapter 1 a structure of projective scheme in the following way: to the ring $k[a, b, c, y]$ give the grading with a, b, c of degree 1 and y of degree 3. Then the equation studied can be written as $F(a, b, c, y) = 0$ with F weighted homogeneous in this grading, and so $S = k[a, b, c, y]/(F)$ will be a graded ring since the ideal (F) is homogeneous. Proj S is the scheme corresponding to the weighted projective variety defined by F.

2.2 Fibred Products, Fibres and Generic Fibres

2.2.1 Definition. If $X \to S$ is a morphism of schemes, we say X is a scheme *over* S.

2.2.2 Definition (Fibred Product)**.** Let S be a scheme and $X \to S$, $Y \to S$ be schemes over S. The *fibred product* of X and Y over S is defined to be the scheme $X \times_S Y$ together with morphisms $X \times_S Y \to X$ and $X \times_S Y \to Y$ that commute with $X \to S$ and $Y \to S$, satisfying the following universal property:

> For every scheme W, every pair of morphisms $W \to X$, $W \to Y$ commuting with $X \to S$ and $Y \to S$ factors uniquely through $X \times_S Y$.

We refer the reader to [Har77], Section II.3, for a proof of the existence and uniqueness of such an object, as well as the proof of the following fundamental fact.

2.2.3 Proposition. *If X, Y, S are affine schemes, say $X =$ Spec A, $Y =$ Spec B and $S =$ Spec R, then the fibred product of X and Y over S is $X \times_S Y =$ Spec $(A \otimes_R B)$.*

For example, if $X =$ Spec $k[x_1, \ldots, x_n]$, $Y =$ Spec $k[y_1, \ldots, y_m]$ and $S =$ Spec k then

$$\begin{aligned} X \times_S Y &= \text{Spec } k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m] \\ &\cong \text{Spec } k[x_1, \ldots, x_n, y_1, \ldots, y_m]. \end{aligned}$$

One of the various uses of the fibred product is to give a scheme structure to the fibres of a morphism:

2.2.4 Definition. Let $f : X \to Y$ be a morphism of schemes, take $y \in Y$ and let $k(y)$ be the *residue field* at y (i.e., the local ring at y modulo its maximal ideal). The *fibre* X_y of f over y is defined to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

2.2.5 Proposition. *With the notation as above, the topological space of the scheme X_y is homeomorphic to the set $f^{-1}(y)$ with its induced topology as a subspace of X.*

Proof. [Har77], Ex. II.3.10. ■

In this way, any morphism of schemes can be viewed as a family of schemes (its fibres) parametrized by the points of the image scheme. This of course gives a nice geometrical setting for morphisms as it allows one to "decompose" a scheme into a family of schemes. Even more, it can actually give *generic* information about the fibres because of the existence of *generic points*. As an illustration consider the following example:

2.2.6 Example. Let $V \subseteq \mathbb{A}^3$ be the variety defined by $z^2 = x^2 - y^2$ and consider the morphism

$$\begin{aligned} \sigma : \quad V \quad &\to \mathbb{A}^1 \\ (x, y, z) &\mapsto z. \end{aligned} \tag{12}$$

The fibre above any $t \in \mathbb{A}^1_k$ is the curve defined in $\mathbb{A}^3_k$ by $t^2 = x^2 - y^2$ and $z = t$, which is trivially isomorphic to the curve $t^2 = x^2 - y^2$ in $\mathbb{A}^2_k$. In this manner, V is composed of conics parametrized by the z-axis, all of them being smooth except when $t = 0$, in which case the fibre consists of two lines intersecting at the origin, see Figure 3.

Now, the scheme associated to the variety V is given by

$$\operatorname{Spec} k[x, y, z]/(z^2 - x^2 + y^2),$$

and σ can be extended to the morphism of schemes

$$\bar{\sigma} : \operatorname{Spec} k[x, y, z]/(z^2 - x^2 + y^2) \to \operatorname{Spec} k[z]$$

as was explained in section 2.1.

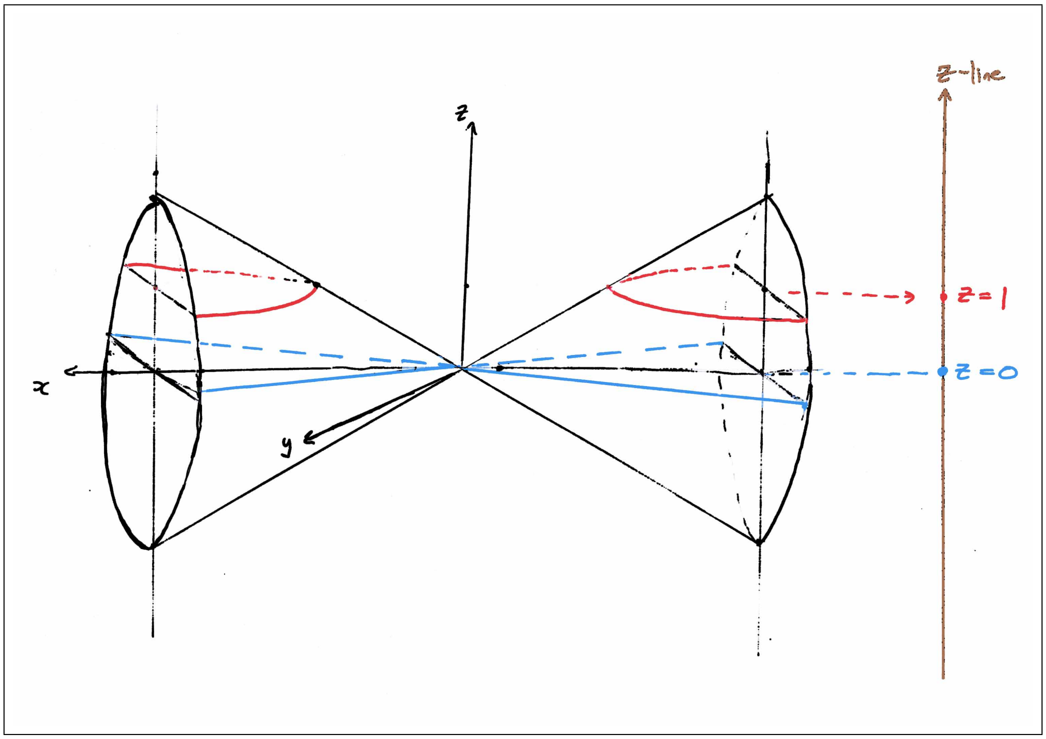

Figure 3: Fibres of σ

In this setting, the fibre of σ above $z = t$ corresponds to the fibre of $\bar{\sigma}$ above the point $(z - t)$. By definition this is given by

$$\text{Spec } k[x, y, z]/(z^2 - x^2 + y^2) \times_{\text{Spec } k[z]} \text{Spec } k((z - t))$$

where $k((z - t))$ is the residue field of the local ring $k[z]_{(z-t)}$. By Proposition 2.2.3 this scheme is the affine scheme

$$\text{Spec } k[x, y, z]/(z^2 - x^2 + y^2) \otimes_{k[z]} k((z - t)).$$

However,

$$k[x, y, z]/(z^2 - x^2 + y^2) \otimes_{k[z]} k((z - t)) \cong k[x, y]/(t^2 - x^2 + y^2)$$

(see Appendix C, Proposition C.2 for a proof), so the fibre of $\bar{\sigma}$ above the point $(z - t)$ is

$$\text{Spec } k[x, y]/(t^2 - x^2 + y^2),$$

that is, the fibre is the scheme corresponding to the hyperbola $t^2 = x^2 - y^2$ in $\mathbb{A}^2_k$ as expected. Note by the way that for $t = 0$ we are constructing the spectrum of a reducible ring since the fibre over $t = 0$ is not irreducible.

There is however an extra point in Spec $k[z]$ which was not present in the classical variety $\mathbb{A}^1_k$: the *generic* point (0) whose closure is all of Spec $k[z]$. The fact that this point is dense suggests the fact that it carries information about almost all $\mathbb{A}^1_k$ and so in some sense the fibre over this point should contain information about almost all the fibres. This will be made precise in what follows.

2.2.7 Definition. Let A be an integral domain and $\psi : X \to \text{Spec } A$ be a morphism of schemes. The fibre of ψ over the point $(0) \in \text{Spec } A$ is called the *generic fibre* of ψ.

For any morphism of varieties we will define its generic fibre as the generic fibre of the corresponding morphism of their associated schemes.

Generic fibres contain information about almost all the fibres. The exact extent of the previous statement is hard to establish, but the main idea is that there are properties (such as smoothness) for which the fact that the generic fibre has the property implies that "almost all" the fibres will have the property (meaning the fibres above an open set). It then comes as no surprise why this fibre is called the generic fibre.

2.2.8 Example. The generic fibre of the map σ in (12) is

$$\text{Spec } k(z)[x, y]/(z^2 - x^2 + y^2)$$

whose classical-algebraic geometry counterpart is the curve $z^2 = x^2 - y^2$ defined in $\mathbb{A}^2_{k(z)}$. This will follow from the next Proposition. Note that $z^2 = x^2 - y^2$ is a smooth curve in $\mathbb{A}^2_{k(z)}$ so all but finitely many fibres are smooth. Of course, we already knew that only the fibre corresponding to $t = 0$ is singular.

2.2.9 Proposition. *Let the hypersurface $V \subseteq \mathbb{A}^n_k$ with coordinates $x_1, \ldots, x_n$ be defined by some $f \notin k[x_i]$ for some fixed i. The generic fibre of the morphism $(x_1, \ldots, x_n) \mapsto x_i$ is*

$$\text{Spec } k(x_i)[x_1, \ldots, \hat{x}_i, \ldots, x_n]/(f)$$

Proof. By definition, the generic fibre is given by

$$\text{Spec } k[x_1, \ldots, x_n]/(f) \times_{\text{Spec } k[x_i]} \text{Spec } k((0))$$

where $k((0))$ is the residue field of the local ring $k[x_i]_{(0)}$, that is, $k((0)) = k(x_i)$. By Proposition 2.2.3 this scheme is the affine scheme

$$\text{Spec } k[x_1, \ldots, x_n]/(f) \otimes_{k[x_i]} k(x_i)$$

and this ring is isomorphic to $k(x_i)[x_1, \ldots, \hat{x_i}, \ldots, x_n]/(f)$, see Appendix C, Proposition C.3 for a proof. ■

The important fact about generic fibres that we will use is that a closed point in the generic fibre of a morphism of a variety over a curve *specializes* to a point in all but finitely many fibres. To illustrate this fact let us return to the example.

Any closed point of the generic fibre Spec $k(z)[x,y]/(z^2 - x^2 + y^2)$ corresponds to a point $P = (x_P, y_P)$ on the associated curve $z^2 = x^2 - y^2$ in $\mathbb{A}^2_{k(z)}$. Therefore $x_P, y_P \in k(z)$ so the coordinates of P are rational functions in z over k, say $P = (\phi(z), \psi(z))$. This implies that the equation

$$z^2 = \phi(z)^2 - \psi(z)^2$$

holds *generically.* Now, the denominators of $\phi(z)$ and $\psi(z)$ only vanish at finitely many $t \in k$, and for any $t \in k$ for which they do not vanish we have

$$t^2 = \phi(t)^2 - \psi(t)^2.$$

This implies that $P_t = (x, y, z) = (\phi(t), \psi(t), t)$, the point obtained by specializing P at t, is a point on the fibre above $z = t$.

Some examples of points on the generic fibre[1] are

$$\begin{aligned} P_1 &= (z, 0) \qquad (13) \\ P_2 &= (-z, 0) \\ P_3 &= (5z/3, 4z/3) \\ P_4 &= \left(\frac{z(z^2+1)}{z^2-1}, \frac{2z^2}{z^2-1}\right). \end{aligned}$$

Note also that both P_3 and P_4 specialize to $(10/3, 8/3, 2) \in V$ for $t = 2$, so every point in V may be specialized to by more than one point on the generic fibre (see Figure 4).

As a matter of fact, we can naturally identify the set of points on the generic fibre with the *sections* of the morphism through the specialization map. We will prove this for a special case in Proposition 2.2.11. We first define the sections of a morphism.

[1] As a matter of fact

$$Q_m = \left(\frac{z(m^2+1)}{m^2-1}, \frac{2zm}{m^2-1}\right)$$

is a point on the generic fibre for any $m \in k(z)$.

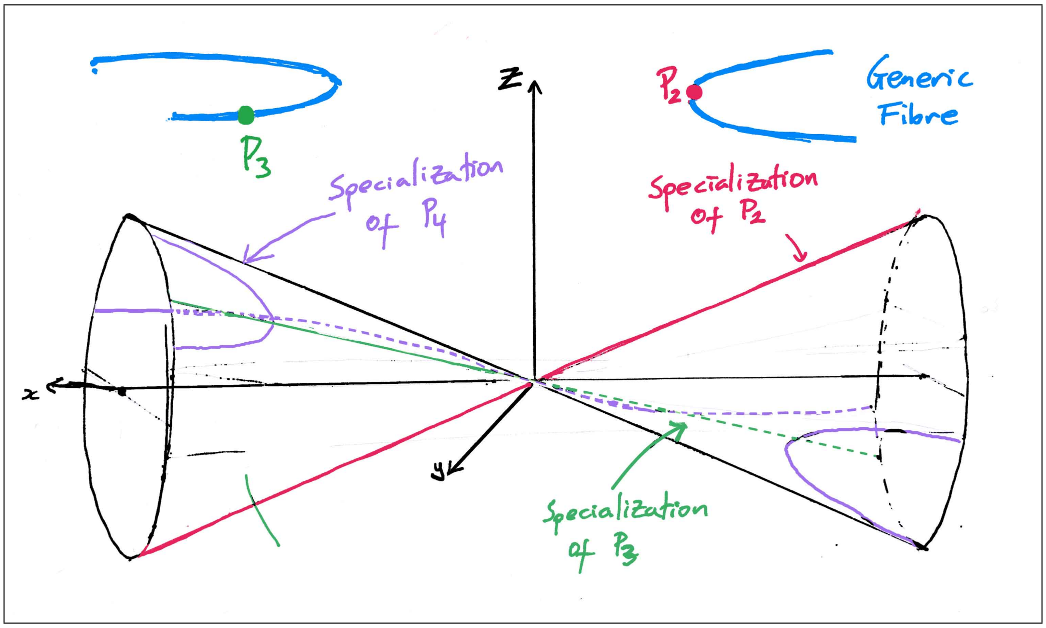

Figure 4: Specialization of points on the generic fibre

2.2.10 Definition. Let $\psi: X \to Y$ be a dominant rational map of varieties. A *section* of ψ is a rational map $\rho: Y \to X$ such that $\psi \circ \rho$ is the identity on Y as a rational map (i.e., on the set where it is defined).

In the case when Y is a curve the image of a section of ψ is simply a curve lying in X that is birationally equivalent to Y, and that maps $1-1$ to Y through ψ.

2.2.11 Proposition. *Let the hypersurface $V \subseteq \mathbb{A}_k^n$ with coordinates $x_1, \ldots, x_n$ be defined by some $f \notin k[x_i]$ for some fixed i. There is a one-to-one correspondence between the set of closed points on the generic fibre and the set of sections of the morphism $\pi : (x_1, \ldots, x_n) \mapsto x_i$.*

Proof. By permutation of the variables we may suppose that $i = n$. By Proposition 2.2.9 the generic fibre of π is given by the scheme

$$\operatorname{Spec} k(x_n)[x_1, \ldots, x_{n-1}]/(f)$$

which corresponds to the hypersurface $H \subseteq \mathbb{A}^{n-1}_{k(x_n)}$ defined by f.

Any closed point on the generic fibre corresponds to a point $P \in H$ and P has coordinates in $k(x_n)$ given by rational functions in x_n over k, say as $x_i(P) = \phi_i(x_n)$ for

$i = 1, \ldots, n-1$. Then, the map defined by $t \mapsto (\phi_1(t), \ldots, \phi_{n-1}(t), t)$ is a section of the projection $\pi : (x_1, \ldots, x_n) \mapsto x_n$.

Conversely, let ρ be a section of π. Then $\rho : \mathbb{A}^1_k \to V$ is given by

$$t \mapsto (\phi_1(t), \ldots, \phi_n(t))$$

on some open subset of $\mathbb{A}^1_k$ where the ϕ_i are rational funtions in t. Since ρ is a section we actually have that $\phi_n(t) = t$. Then $P_\rho = (\phi_1(x_n), \ldots, \phi_{n-1}(x_n))$ is a point on H and therefore gives a closed point on the generic fibre. ■

Therefore, every closed point P on the generic fibre of a morphism such as those encountered in Proposition 2.2.11 gives a curve on V which is birationally equivalent to $\mathbb{A}^1_k$, namely, the image of the corresponding section. This image is the set of the specializations of the point P on all the fibres it specializes to. Conversely, any curve lying inside V that is birationally equivalent to $\mathbb{A}^1_k$ and that maps $1-1$ through π corresponds to a point on the generic fibre. To see why this is so, note that if $C \subset V$ is a curve that is birationally equivalent to $\mathbb{A}^1_k$, then there exists a rational map $t \mapsto (\phi_1(t), \ldots, \phi_n(t)) \in C$. If C maps $1-1$ through π, then ϕ_n is an injective rational function on the set where it is defined. We can therefore set $s = \phi_n(t)$ and reparametrize C on an appropriate set in terms of s by $s \mapsto (\phi_1(\phi_n^{-1}(s)), \ldots, \phi_{n-1}(\phi_n^{-1}(s)), s)$. Then $(\phi_1(\phi_n^{-1}(x_n)), \ldots, \phi_{n-1}(\phi_n^{-1}(x_n)))$ will be a point on the generic fibre. We will sometimes refer to these curves as sections since we can identify them, see Figure 5.

2.2.12 Example. All the points on the generic fibre from the example given in (13) are clearly seen to be sections. Conversely, lines L on the cone can be parametrized as (at, bt, t) with $t \in k$ where $(a, b, 1)$ is the direction vector of L. Then (az, bz) is a point on the generic fibre.

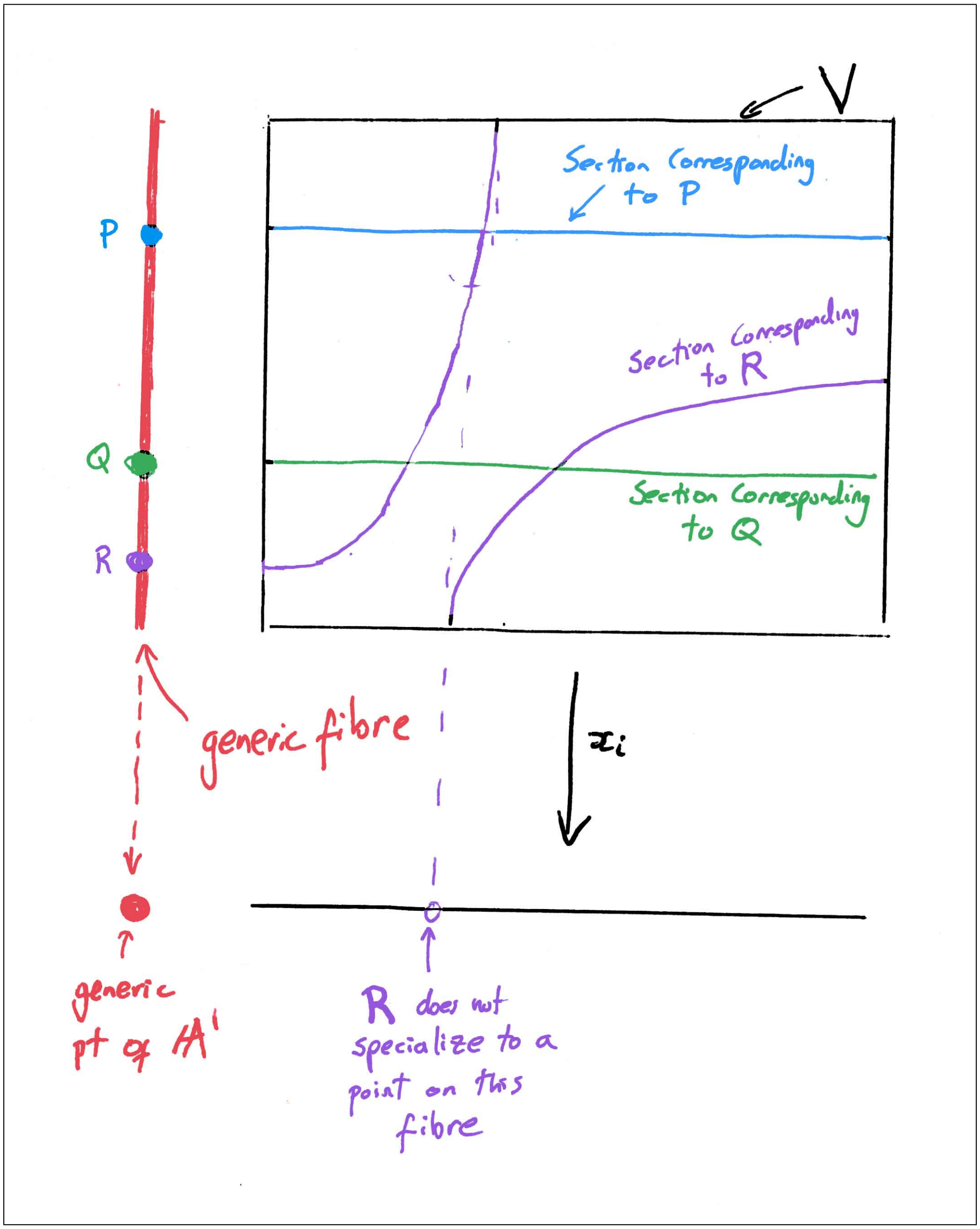

Figure 5: Sections and Points on the Generic Fibre

2.3 Chapter 1 Revisited

To illustrate how the previous concepts clarify the situation encountered in Chapter 1 we will go over it again to illustrate where the main ideas came from. We will mainly focus on Case 5 for which we have all the relevant equations. Of course, the same could be done for Case 6, but the main purpose is to illustrate the way in which any situation as those encountered in Chapter 1 can be analyzed more clearly and intuitively in this broader context.

For Case 5 we were studying the equation $F(a,b,c,y) = 0$ where

$$F = y^2 + (a^2 + 2b^2 - c^2)(a^2 - b^2 - ac)(a^2 - b^2 + ac),$$

which was mentioned to be the defining equation of a surface S in weighted projective space $\mathbb{P}(1,1,1,3)$. This surface was seen to correspond to the projective scheme Proj $\mathbb{Q}[a,b,c,y]/(f)$ in section 2.1.

Our focus, however, was on the affine part S_a of S corresponding to $a \neq 0$ defined by

$$S_a : \quad y_1^2 = (1 + 2\lambda^2 - x^2)(x - (1 - \lambda^2))(x + (1 - \lambda^2)) \tag{14}$$

in $\mathbb{A}^3$ with coordinates $(x, \lambda, y_1) = (c/a, b/a, y/a^3)$. Rewriting this equation as $f(x, \lambda, y_1) = 0$ (note that $f(x, \lambda, y_1) = F(1, \lambda, x, y_1)$) we can regard S_a as the affine scheme Spec $\mathbb{Q}[x, \lambda, y_1]/(f)$ which can be seen to be one of the affine schemes of the standard cover of the projective scheme associated to S.

For S_a we defined a projection morphism

$$\begin{aligned} \sigma : \quad & S_a && \to \mathbb{A}^1 \\ & (x, \lambda, y_1) && \mapsto \lambda \end{aligned}$$

whose generic fibre by Proposition 2.2.9 is given by Spec $\mathbb{Q}(\lambda)[x, y_1]/(f)$, corresponding to the curve E_λ defined by f in the affine plane over the field $\mathbb{Q}(\lambda)$. This curve was mentioned to be an elliptic curve in the proof of Theorem 1.2.1. The fact that the generic fibre was elliptic implied that all but finitely many fibres of σ were elliptic as was seen explicitly.

This of course gives a nice geometric setting to the arithmetic problem that is being studied: A surface has been replaced by a curve (the generic fibre), and each point on this curve corresponds to a curve lying inside the surface (the image of the corresponding section). Moreover, we have a group structure on the generic fibre. Therefore, by starting off with a curve with rational points corresponding to a point on the generic fibre, we have a way to obtain non-trivial curves containing rational points on the surface since multiples of this point give curves on the surface parametrized over $\mathbb{Q}$.

The change of coordinates transforming the equation of the generic fibre into Weierstrass form is obtained by sending the point $O = (x, y_1) = (\lambda^2 - 1, 0)$ to infinity, giving the equation

$$E_\lambda : \quad v^2 = u^3 + A(\lambda)u + B(\lambda) \tag{15}$$

where

$$\begin{aligned} A(\lambda) &= -(16 - 32\lambda^4 + 24\lambda^6 + \lambda^8)/3 \\ B(\lambda) &= 2(2 + \lambda^4)(-32 + 112\lambda^4 - 72\lambda^6 + \lambda^8)/27. \end{aligned}$$

If we take a look at what this change of coordinates does to S_a, then we find a surface E in $\mathbb{A}^3$ with coordinates (u, v, λ) whose defining equation is the same as that of E_λ. This is the surface E that was defined in Chapter 1, which is birationally equivalent to S_a. The point $O = (x, y_1) = (\lambda^2 - 1, 0)$ on the generic fibre corresponds to the section $\rho : \lambda \mapsto (\lambda^2 - 1, \lambda, 0)$ of σ, and the change of coordinates on the generic fibre sending the point to infinity corresponds to the change of variables on S_a sending the image of that section to infinity, giving the surface E. The change of variables puts each fibre in Weierstrass form (see Figure 6).

With the generic fibre in Weierstrass form, we studied the rational points on S. In section 1.2.2 we gave parametrisations of two curves lying inside S, each with infinitely many rational points. We will deal here with C_1 which will call C in this section. The curve C is defined by

$$\begin{aligned} a(t) &= t^2 + 2 \\ b(t) &= 4t \\ c(t) &= a(t) \\ y(t) &= 32t^2(2 - t^2) \end{aligned}$$

and it has an affine part C_a lying inside S_a given by

$$x(t) = 1, \qquad y_1(t) = \frac{32t^2(2 - t^2)}{t^2 + 2}, \qquad \lambda(t) = \frac{4t}{t^2 + 2}. \tag{16}$$

Proposition 2.2.11 implies that C_a does not correspond to a point on the generic fibre E_λ of σ since for all but finitely many λ there are two points in C_a mapping to λ through

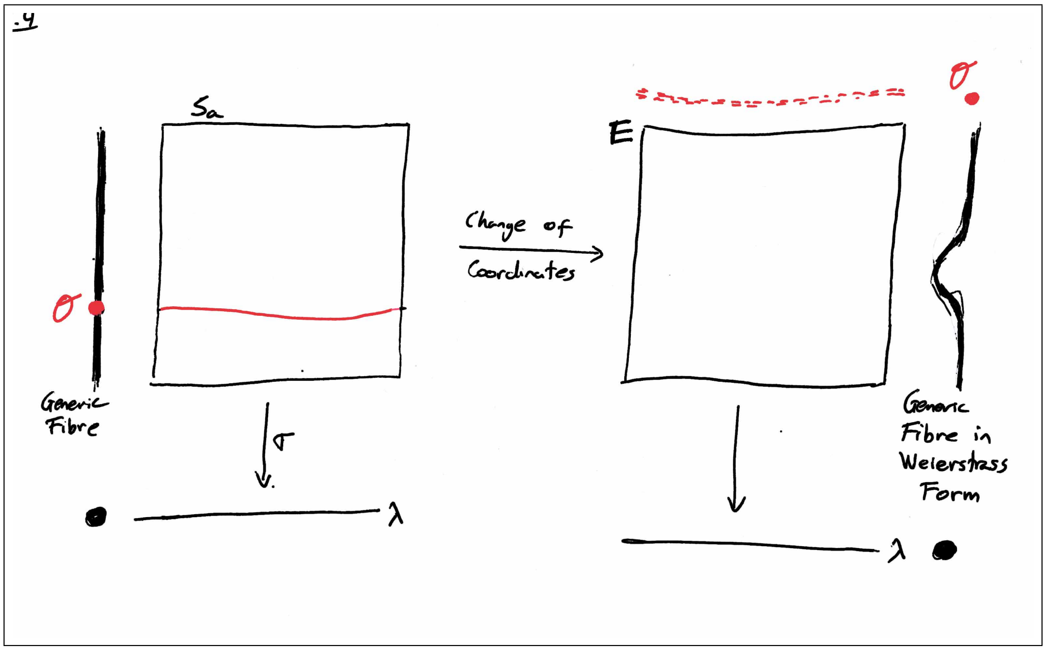

Figure 6: Change of Coordinates

σ (in fact, $\lambda(t) = \lambda(2/t)$). However, it is readily seen that C_a corresponds to a point on the generic fibre over a field extension of $\mathbb{Q}(\lambda)$. As a matter of fact, the equation

$$\lambda = \frac{4t}{t^2 + 2}$$

implies that $t^2 - (4/\lambda)t + 2 = 0$, and so $\mathbb{Q}(t)$ is a quadratic field extension of $\mathbb{Q}(\lambda)$.

An equation for E_λ over the field $\mathbb{Q}(t)$ is obtained by replacing λ by $4t/(t^2+2)$ in (14), and the curve C_a corresponds to the point $(x(t), y_1(t))$ coming from (16). Note that the restriction of σ to C_a is a $2-1$ map. As a matter of fact, every point on E_λ over $\mathbb{Q}(t)$ corresponds to a curve in S_a that maps at most $2-1$ to the λ-line through σ. Specifically, if $(\phi(t), \psi(t))$ is a point in E_λ over $\mathbb{Q}(t)$, then the equality in (14) holds after replacing $\phi(t)$ for x, $\psi(t)$ for y_1, and λ for $4t/(t^2+2)$. The image of the map $t \mapsto (\phi(t), 4t/(t^2+2), \psi(t))$ is therefore a curve in S_a.

To compute multiples of the point corresponding to C_a we use an equation in Weierstrass form, which we already have found over $\mathbb{Q}(\lambda)$ in (15) so it is simply a matter of replacing λ by $4t/(t^2+2)$ in (15). The resulting equation is precisely the one given in (7) in Chapter 1, and C_a corresponds in the new coordinates (u, v) to the point P_C given in the same section.

Corollary 1.2.3 proves that P_C has infinite order in $E_\lambda(\mathbb{Q}(t))$ and this implies that the subgroup generated by C_a in $E_\lambda(\mathbb{Q}(t))$ corresponds to an infinite family of curves lying in S_a each mapping $2-1$ to the λ-line through σ, and parametrized by rational functions in $\mathbb{Q}(t)$. See figure 7.

This was seen in 1.2.5 to imply that the set of rational points in S_a is Zariski dense.

Regarding the geometry behind the situation, it should be noted that it would have been convenient (geometrically) to work with projective coordinates under which σ takes the form[2]

$$\begin{aligned} \sigma : \quad & S && \to \mathbb{P}^1 \\ & [a:b:c:y] && \mapsto [a:b] \end{aligned}$$

but the complications of dealing in depth with projective schemes refrained us from engaging on writing the theory in this setting with the time restrains we had. Note that σ would no longer be a morphism as there is a point $[a:b:c:y] = [0:0:1:0]$ where it is not defined. To define the fibration everywhere we have to blow up the surface at that point.

[2]Remember that $\lambda = b/a$

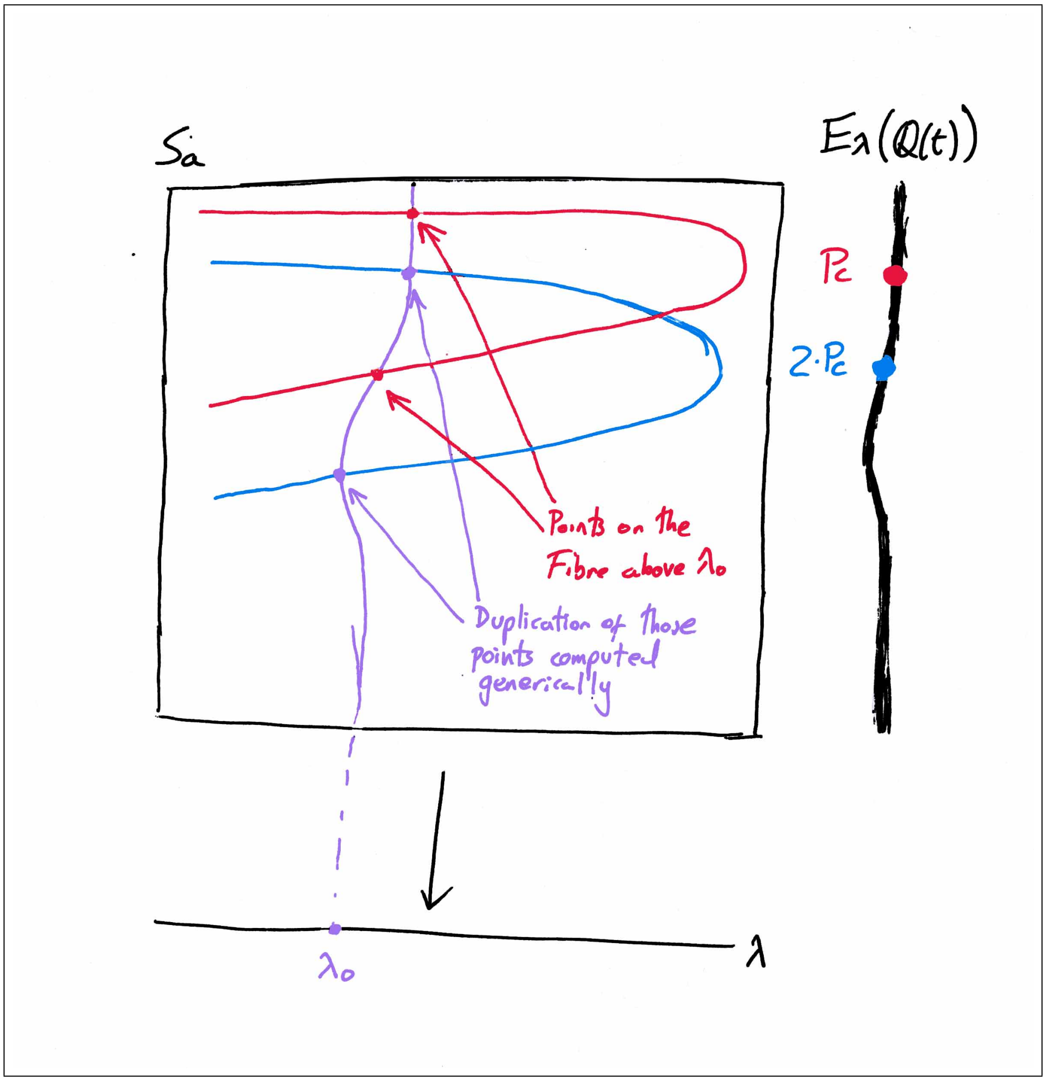

Figure 7: Points on $E_\lambda(\mathbb{Q}(t))$ correspond to curves in S_a

Chapter III

K3 SURFACES

In this chapter we will take a closer look at weighted projective surfaces defined by equations of the form $Y^2 = F(A, B, C)$ in $\mathbb{P}(1, 1, 1, 3)$ with coordinates[1] A, B, C, Y such as those related to 3-parameter tetrahedra. Our focus will be on smooth surfaces of this form, which implies that the results will not be directly applicable to the surfaces related to tetrahedra as these are all singular. We will prove that any smooth surface in $\mathbb{P}(1, 1, 1, 3)$ given by an equation of the form $Y^2 = F(A, B, C)$ is a $K3$ surface. As will be seen, and even though very little is known about the arithmetic of $K3$ surfaces, this will give some insight on the arithmetic properties that could be expected from the surfaces related to 3-parameter tetrahedra.

3.0.1 Theorem. *Let X be a smooth weighted projective surface defined by*

$$X: \quad Y^2 = F(A, B, C)$$

in weighted projective space $\mathbb{P}(1, 1, 1, 3)$ with coordinates A, B, C, Y. Then $K_X = 0$, where K_X is the canonical class in the Picard Group of X.

Proof. We will prove that the divisor of the 2-form

$$\omega = \frac{A^3}{Y}\, d\left(\frac{B}{A}\right) \wedge d\left(\frac{C}{A}\right)$$

is the zero divisor. We first take a look at the affine part of the surface $A \neq 0$ using affine coordinates $b = B/A$, $c = C/A$, $y = Y/A^3$. The equation for this affine part of X is given by

$$X_A: \quad y^2 = f(b, c)$$

[1] We will use capital letters for the projective coordinates since we will be using affine coordinates extensively.

where $f(b,c) = F(1,b,c)$, and ω takes the form

$$\omega = \frac{1}{y}\, db \wedge dc.$$

Let $P = (b,c,y) = (r,s,t)$ be a point on X_A. There are two cases to consider depending whether or not t is equal to 0.

If $t \neq 0$, then by the equation for X_A we have

$$\begin{aligned} y^2 - t^2 &= f(b,c) - t^2 \\ &= (f(b,c) - f(r,s)) + (f(r,s) - t^2) \\ &= f(b,c) - f(r,s) \end{aligned}$$

since P is a point of X_A and so $f(r,s) - t^2 = 0$. The expression in the last equality is in the ideal $\langle b-r,\, c-s\rangle$ since it vanishes when $b = r$ and $c = s$. Since $t \neq 0$, we can invert $y+t$ in the local ring at P, and since $y^2 - t^2 = (y+t)(y-t)$, it follows that $y - t \in \langle b-r,\, c-s\rangle$ in the local ring at P. This implies that $b-r$ and $c-s$ are local parameters at P as the maximal ideal at P is generated by $b-r$, $c-s$ and $y-t$, and therefore, by the above, by just $b-r$ and $c-s$. Therefore

$$\omega = \frac{1}{y}\, db \wedge dc = \frac{1}{y}\, d(b-r) \wedge d(c-s)$$

has no poles or zeros at P.

If $t = 0$ then $f(r,s) = 0$, and the smoothness of X_A implies that one of $f_b(r,s)$ or $f_c(r,s)$ is non zero. Suppose without loss of generality that $f_c(r,s) \neq 0$. By the equation for X_A we have

$$\begin{aligned} y^2 &= f(b,c) \\ &= f(b,c) - f(r,s) \\ &= (b-r)g(b,c) + (c-s)h(b,c) \end{aligned}$$

for some $g,h \in k[b,c]$ as $f(b,c) - f(r,s)$ vanishes when $b = r$ and $c = s$. Moreover, it can bee seen that $h(r,s) = f_c(r,s) \neq 0$ by differentiating the equation $f(b,c) - f(r,s) = (b-r)g(b,c) + (c-s)h(b,c)$ with respect to c and evaluating at (r,s). This implies that y and $b-r$ are local parameters at P since $c - s \in \langle y,\, b-r\rangle$ in the local ring at P as $h(b,c)$ does not vanish at P and so is invertible. Moreover, by the equation of X_A we have

$$2ydy = d(f(b,c)) = f_b db + f_c dc$$

and so,

$$dc = \frac{2y}{f_c}\, dy - \frac{f_b}{f_c}\, db.$$

Therefore,

$$\omega = \frac{1}{y}\, db \wedge dc = \frac{2}{f_c}\, db \wedge dy = \frac{2}{f_c}\, d(b-r) \wedge dy$$

which implies that ω has no poles or zeros at P in this case either.

The previous computations imply that there is no contribution to the divisor of ω in the affine part $A \neq 0$ as ω has no zeros or poles at any point.

To finish the proof we have to see that there is no contribution of $A = 0$, and for this we take a look at another affine part of X. Let X_B be the affine part of X of points satisfying $B \neq 0$. The affine part of $A = 0$ in X_B is given by $a' = 0$, and the equation of X_B is given by

$$X_B : \quad y'^2 = g(a', c')$$

with affine coordintes $a' = A/B$, $c' = C/B$, $y' = Y/B^3$ where $g(a', c') = F(a', 1, c')$. In this affine part ω takes the form

$$\omega = \frac{a'^3}{y'}\, d\left(\frac{1}{a'}\right) \wedge \left(\frac{c'}{a'}\right) = \frac{-1}{y'}\, da' \wedge dc',$$

since $d(1/a') = -(1/a')^2 da'$ and $d(c'/a') = (1/a')^2(a'dc' - c'da')$.

Let $P = (a', c', y') = (0, s, t)$ be a point on X_B with $A = 0$. From the equation of X_B we have

$$\begin{aligned} y'^2 - t^2 &= g(a', c') - t^2 \\ &= (g(a', c') - g(0, s)) + (g(0, s) - t^2) \\ &= g(a', c') - g(0, s), \end{aligned}$$

where last expresion is in the ideal $\langle a',\, c' - s\rangle$ since it vanishes for $a' = 0$ and $c' = s$. This implies that if $t \neq 0$ then $y' - t \in \langle a',\, c' - s\rangle$ in the local ring at P since $y + t$ is invertible, and so a' and $c' - s$ are local parameters at P. Moreover, since there are at most deg g points on $a' = 0$ satisfying $y' = 0$, we have found local parameters for most of the points on the line defined by $a' = 0$ in X_B. Since

$$\omega = \frac{1}{y'}\, da' \wedge dc' = \frac{1}{y'}\, da' \wedge d(c' - s),$$

this implies that ω has no poles or zeros for any P on $a' = 0$ with $t \neq 0$ and so there is no contribution of $A = 0$ to the divisor of ω.

Therefore, $K_X = 0$ as stated. ■

The fact that $K_X = 0$ implies that the Kodaira dimension of a smooth projective surface in $\mathbb{P}(1,1,1,3)$ given by an equation of the form $Y^2 = F(A,B,C)$ is zero. Since the Kodaira dimension is a birational invariant, and the Kodaira dimension of $\mathbb{P}^2$ is -1, we obtain the following corollary.

3.0.2 Corollary. *If X is a smooth projective surface given by an equation of the form $Y^2 = F(A,B,C)$ in weighted projective space $\mathbb{P}(1,1,1,3)$, then X is not parametrizable.*

This gives some geometric reasons to believe that surfaces related to 3-parameter tetrahedra are not parametrizable, even though none of these surfaces are smooth, so there is some chance they could be. In fact, as was stated in the introduction, the surfaces related to cases 2 and 8 have been proved to be parametrizable over $\mathbb{Q}$. The nature of the singularities of these surfaces therefore plays a very important role.

If K_Y can be proved to be zero for a minimal non singular model Y of the surfaces of cases 5 and 6, this would imply that they are not parametrizable, so in some sense the results obtained in Chapter 1 are the best one can obtain regarding the nature of the set of its rational points (namely, that the set of rational points is Zariski dense). The nature of the singularities of the surfaces for cases 5 and 6 seems to indicate that this is in fact the case. This could be proved by finding a non singular model Y of the surfaces by blowing up the surfaces at the singular points and then checking that there is no contribution to K_Y coming from the exceptional divisors. However, these computations will not be undertaken in this thesis.

We will finish our analysis of smooth surfaces of the form $Y^2 = F(A,B,C)$ by proving that they are $K3$ surfaces. We have already done part of the work as by definition a $K3$ surface is a smooth projective irreducible surface for which $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$. We begin with a lemma.

3.0.3 Lemma. *The Euler characteristic $\chi(\mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ of a smooth projective surface X given by an equation of the form $Y^2 = F(A,B,C)$ in $\mathbb{P}(1,1,1,3)$ is 2.*

Proof. We will prove this with the use of the Hilbert polynomial of X (see [Har77], Section I.7). It is known that if $H(d)$ is the Hilbert polynomial of a smooth projective variety, the Euler charactaristic of the structure sheaf of the variety is equal to $H(0)$. In the graded ring $k[A,B,C,Y]/\langle Y^2 - F(A,B,C)\rangle$ with A, B, C of weight 1 and Y of weight 3 every element can be expressed express as

$$G(A,B,C)+Y\cdot H(A,B,C)$$

where G is homogeneous of degree d and H is homogeneous of degree $d-3$ since we can replace all the even powers of Y by an appropriate power of $F(A,B,C)$. Moreover, since the dimension as a k-vector space of the homogeneous polynomials of degree d in the variables A,B,C can be seen to be equal to $f(d)=\binom{d+2}{2}$, we obtain that the dimension of the weighted homogeneous elements of weight $d>3$ in $k[A,B,C,Y]/\langle Y^2-F(A,B,C)\rangle$ is given by

$$f(d)+f(d-3)=d^2+2.$$

Therefore, $H(d)=d^2+2$ and $\chi(\mathcal{O}_X)=H(0)=2$ as stated. ■

3.0.4 Theorem. *Any smooth projective surface X given by an equation of the form $Y^2=F(A,B,C)$ is a $K3$ surface.*

Proof. It only remains to prove that $H^1(X,\mathcal{O}_X)=0$. For this we will prove that $\dim_k H^1(X,\mathcal{O}_X)=0$. Let $h^i(\mathcal{F})=\dim_k H^i(X,\mathcal{F})$ for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, let $\chi(\mathcal{O}_X)$ be the Euler characteristic of the structure sheaf $\mathcal{O}_X$ of X, and let ω_X be the canonical sheaf of X. We have,

$$\begin{aligned}\chi(\mathcal{O}_X) &= \sum_{i=0}^{\infty}(-1)^i h^i(\mathcal{O}_X)\\ &= h^0(\mathcal{O}_X)-h^1(\mathcal{O}_X)+h^2(\mathcal{O}_X)\\ &= h^0(\mathcal{O}_X)-h^1(\mathcal{O}_X)+h^0(\omega_X)\\ &= 2h^0(\mathcal{O}_X)-h^1(\mathcal{O}_X)\\ &= 2-h^1(\mathcal{O}_X)\end{aligned}$$

where the second equality follows form Grothendiek's vanishing theorem, the third form Serre Duality, the fourth from the fact that $K_X=0$ as this implies that $\omega_X\cong\mathcal{O}_X$, and the fifth since X is connected and projective which implies $h^0(\mathcal{O}_X)=1$. By Lemma 3.0.3 we have $\chi(\mathcal{O}_X)=2$, and so $h^1(\mathcal{O}_X)=0$. ■

Appendix A

CURVES OF THE FORM $Y^2 = A + BX + CX^2 + DX^3 + EX^4$

A.1 Theorem. *Any curve defined over k by an equation of the form $y^2 = f(x)$ with $f(x) \in k[x]$ of degree 4 with a root in k and no repeated roots over $\bar{k}$ is an elliptic curve over k, and the change of variables*

$$\begin{aligned} u &= \frac{\beta}{x-\alpha} \\ v &= u^2 y = \frac{\beta^2 y}{(x-\alpha)^2} \end{aligned}$$

where $\alpha, \beta \in k$, $f(\alpha) = 0$ and $\beta \neq 0$ transforms the equation of the curve to $v^2 = g(u)$ with $g \in k[u]$ of degree 3 with no repeated roots over $\bar{k}$.

Proof. Since $f(\alpha) = 0$ we have

$$f(x) = a_1(x-\alpha) + a_2(x-\alpha)^2 + a_3(x-\alpha)^3 + a_4(x-\alpha)^4$$

for some $a_i \in k$. Now, $u = \beta/(x-\alpha)$ implies $x = \alpha + \beta/u$ and so

$$\begin{aligned} y^2 &= f(\alpha + \beta/u) \\ &= a_1(\beta/u) + a_2(\beta/u)^2 + a_3(\beta/u)^3 + a_4(\beta/u)^4. \end{aligned}$$

Multiplying this equation by u^4 we obtain

$$(u^2y)^2 = a_1\beta u^3 + a_2\beta^2u^2 + a_3\beta^3u + a_4\beta^4,$$

and finally setting $v = u^2y$ gives

$$v^2 = g(u) = \beta a_1 u^3 + a_2\beta^2u^2 + a_3\beta^3u + a_4\beta^4.$$

To see that the curve is elliptic we only have to check that g has no repeated roots. By hypothesis, f has no repeated roots and so f and f' have no common root. Therefore, since $g(u) = u^4 f(\alpha + \beta/u)$, we have $g'(u) = u^2(4uf(\alpha+\beta/u) - f'(\alpha+\beta/u))$. Now, $u = 0$ is not a root of g as otherwise $a_4 = 0$ and this would contradict the asumption that f is of degree 4. If γ is a root of $g(u)$ then $f(\alpha + \beta/\gamma) = 0$ and the expresion for $g'(u)$ implies $g'(\gamma) = \gamma^2 f'(\alpha + \beta/\gamma) \neq 0$ since f and f' have no common roots. Therefore, g has no repeated roots. ■

Note from the proof that by the choosing β cleverly we can make the resulting cubic in u be monic, giving an equation in Weierstrass form.

Appendix B

TRANSFORMATION OF CASE 6

The surface from Case 6 is given by

$$y^2 = 2(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2).$$

Setting $\lambda = c/(a-b)$, $x_1 = a/b$ and $y_1 = y/b^2(a-b)$ we obtain

$$y_1^2 = f(x_1)$$

where $f(x_1) \in \mathbb{Q}(\lambda)[x_1]$ is given by

$$f(x_1) = 2[x^2(1-\lambda^2) + 2\lambda^2 x_1 + 1 - \lambda^2][x_1(1+\lambda^2) - (\lambda^2 - 1)][x_1(\lambda^2 - 1) - (\lambda^2 + 1)]$$

The change of variables

$$x_2 = \frac{1}{x_1 - \alpha}, \quad y_2 = x_2^2 y_1$$

where $\alpha = (\lambda^2 - 1)/(\lambda^2 + 1)$ taking the point $(x_1, y_1) = (\alpha, 0)$ to infinity as seen in Appendix A gives an equation of the form

$$y_2^2 = f_2(x_2) \tag{17}$$

where $f_2(x_2) = x_2^4 f_1(\alpha + \beta/x_2)$ is of degree 3 with leading coeficient

$$c = -16\lambda^2(\lambda^2 - 1)^2/(\lambda^2 + 1)^2.$$

To remove this coefficient multiply (17) by $(\lambda^2 + 1)^6 c^2$ and rewrite in terms of the new variables

$$x_3 = (\lambda^2 + 1)^2 c x_2, \quad y_3 = (\lambda^2 + 1)^3 c y_2$$

Finally, renaming $u = x_3$ and $v = y_3$ gives

$$v^2 = u^3 + a_1(\lambda)u^2 + a_2(\lambda)u + a_3(\lambda)$$

where

$$
\begin{aligned}
a_1(\lambda) &= 2^2(\lambda^2+1)(\lambda^6-15\lambda^4+7\lambda^2-1) \\
a_2(\lambda) &= -2^2(\lambda^2-1)(5\lambda^2-1)(\lambda^2+1)^4 \\
a_3(\lambda) &= -2^9(\lambda^2-1)^2(\lambda^2+1)^7.
\end{aligned}
$$

Appendix C

SOME RESULTS FROM COMMUTATIVE ALGEBRA

C.1 Proposition. *Let R be a commutative ring with identity, S_1 and S_2 be R-algebras, and I_1, I_2 be ideals of S_1 and S_2 respectively. Then*

$$S_1/I_1 \otimes_R S_2/I_2 \cong (S_1 \otimes_R S_2)/I$$

where $I = I_1 \otimes_R S_2 + S_1 \otimes_R I_2$ is the ideal generated by $I_1 \otimes_R S_2$ and $S_1 \otimes_R I_2$ in $S_1 \otimes_R S_2$.

Proof. We will drop the subscript from $\otimes_R$. The map

$$\begin{aligned} S_1/I_1 \times S_2/I_2 &\to (S_1 \otimes S_2)/I \\ ([s_1],[s_2]) &\mapsto [s_1 \otimes s_2] \end{aligned}$$

is well defined since if $[s_1] = [t_1]$ and $[s_2] = [t_2]$ then $t_k = s_k + i_k$ where $i_k \in I_k$ for $k = 1, 2$. Therefore,

$$\begin{aligned} ([t_1],[t_2]) &\mapsto [(s_1 + i_1) \otimes (s_2 + i_2)] \\ &= [s_1 \otimes s_2 + s_1 \otimes i_2 + i_1 \otimes s_2 + i_1 \otimes i_2] \\ &= [s_1 \otimes s_2] \end{aligned}$$

as each of the other terms belong to I. Since this map is also R-bilinear, it induces a homomorphism of R-modules

$$\begin{aligned} \sigma : S_1/I_1 \otimes S_2/I_2 &\to (S_1 \otimes S_2)/I \\ [s_1] \otimes [s_2] &\mapsto [s_1 \otimes s_2] \end{aligned}$$

that is also an R-algebra homomorphism.

Conversely, the map defined by

$$\begin{aligned} S_1 \times S_2 &\to S_1/I_1 \otimes S_2/I_2 \\ (s_1, s_2) &\mapsto [s_1] \otimes [s_2] \end{aligned}$$

is R-bilinear, so it induces an R-module homomorphism

$$\begin{aligned} \phi : S_1 \otimes S_2 &\to S_1/I_1 \otimes S_2/I_2 \\ s_1 \otimes s_2 &\mapsto [s_1] \otimes [s_2] \end{aligned}$$

that is also a ring homomorphism. Since $I \subset \text{Ker } \phi$, there is an induced homomorphism

$$\begin{aligned} \hat{\phi} : (S_1 \otimes S_2)/I &\to S_1/I_1 \otimes S_2/I_2 \\ [s_1 \otimes s_2] &\mapsto [s_1] \otimes [s_2]. \end{aligned}$$

The isomorphism follows as σ and $\hat{\phi}$ are inverses to each other. ∎

C.2 Proposition. *Let $f \in k[x_1, \ldots, x_n]$ and $t \in k$. Then*

$$k[x_1, \ldots, x_n]/(f) \otimes_{k[x_n]} k[x_n]_{(x_n - t)}/(x_n - t) \cong k[x_1, \ldots, x_{n-1}]/(f(x_1, \ldots, x_{n-1}, t))$$

where $k[x_n]_{(x_n-t)}$ is the localization of $k[x_n]$ at $(x_n - t)$.

Proof. We will prove the result for $n = 2$, the proof of general case is similar. Let $f \in k[x, y]$ and consider the map $k[y] \to k[y]_{(y-t)}/(y - t)$ sending $p(y) \mapsto [p(y)/1]$. The kernel of this map is $(y - t)$ since $(y - t)$ is contained in the kernel, and $(y - t)$ and is a maximal proper ideal so it equals the kernel. Therefore,

$$k[y]/(y - t) \cong k[y]_{(y-t)}/(y - t)$$

as $k[y]$-algebras by the map induced by $k[y] \to k[y]_{(y-t)}/(y - t)$ and so,

$$k[x, y]/(f) \otimes_{k[y]} k[y]_{(y-t)}/(y - t) \cong k[x, y]/(f) \otimes_{k[y]} k[y]/(y - t).$$

We will drop the subscript of $\otimes_{k[y]}$ form now on. By Proposition C.1, the tensor product on the right is isomorphic to

$$(k[x, y] \otimes k[y])/I$$

where I is the ideal generated by $f \otimes 1$ and $1 \otimes (y - t) = (y - t) \otimes 1$ in $k[x, y] \otimes k[y]$. However, $k[x, y] \otimes k[y] \cong k[x, y]$ and so

$$(k[x, y] \otimes k[y])/I \cong k[x, y]/(f, y - t).$$

Consider now the surjective map

$$\begin{aligned}\phi: \quad k[x,y] &\to k[x]/(f(x,t)) \\ p(x,y) &\mapsto [p(x,t)].\end{aligned}$$

Clearly $(f, y-t) \subset \text{Ker } \phi$. Conversely, if $p(x,y) \in \text{Ker } \phi$, then $p(x,t) = h(x)f(x,t)$ for some $h(x) \in k[x]$ and so

$$p(x,y) - h(x)f(x,y) \in k[x,y]$$

vanishes at $y = t$. This implies that $p(x,y) - h(x)f(x,y) \in (y-t)$, that is, $p(x,y) \in (f, y-t)$. Therefore, Ker $\phi = (f, y-t)$ and so

$$k[x,y]/(f, y-t) \cong k[x]/(f(x,t))$$

as stated. ■

C.3 Proposition. *Let $f \in k[x_1, \ldots, x_n]$. There is an isomorphism*

$$k[x_1, \ldots, x_n]/(f) \otimes_{k[x_n]} k(x_n) \cong k(x_n)[x_1, \ldots, x_{n-1}]/(f).$$

Proof. As with the previous proof, let $n = 2$. The proof of general case is similar. Let the $f \in k[x,y]$ with $f \notin k[y]$. The map

$$\begin{aligned}\phi: \quad k[x,y]/(f) \times k(y) &\to k(y)[x]/(f) \\ ([p(x,y)], \phi(y)) &\mapsto [p(x,y)\phi(y)].\end{aligned}$$

is well defined and bilinear over $k[y]$. Therefore it induces a $k[y]$-algebra homomorphism

$$\begin{aligned}\phi: \quad k[x,y]/(f) \otimes k(y) &\to k(y)[x]/(f) \\ [p(x,y)] \otimes \phi(y) &\mapsto [p(x,y)\phi(y)],\end{aligned}$$

Now, every element in $k(y)[x]$ can be written in the form $q(x,y)/r(y)$ for some $q(x,y) \in k[x,y]$ and $r(y) \in k[y]$, and every element in $k[x,y]/(f) \otimes k(y)$ can be written in the form $[p(x,y)] \otimes 1/h(y)$ for some $p(x,y) \in k[x,y]$ and $h(y) \in k[y]$. If $[p(x,y)] \otimes 1/h(y) \in \text{Ker } \phi$, then

$$\frac{p(x,y)}{h(y)} = \frac{q(x,y)}{r(y)} f(x,y)$$

for some $q(x,y) \in k[x,y]$ and $r(y) \in k[y]$ and so,

$$r(y)p(x,y) = h(y)q(x,y)f(x,y)$$

in $k[x,y]$. Therefore,

$$\begin{aligned}
[p(x,y)] \otimes \frac{1}{h(y)} &= [p(x,y)] \otimes \frac{r(y)}{r(y)h(y)} \\
&= [r(y)p(x,y)] \otimes \frac{1}{r(y)h(y)} \\
&= [h(y)q(x,y)f(x,y)] \otimes \frac{1}{r(y)h(y)} \\
&= [0] \otimes \frac{1}{r(y)h(y)} \\
&= 0
\end{aligned}$$

and so ϕ is injective. Since ϕ is also surjective,

$$k[x,y]/(f) \otimes k(y) \cong k(y)[x]/(f)$$

as stated. ■

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