Quadratic Forms
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Definition 1. A *Quadratic Form* over a ring R is a function

$$Q(\vec{x}) = \sum c_{ij} x_i x_j$$

with $c_{ij} \in R$. We can write this as

$$Q(\vec{x}) = \frac{1}{2} \vec{x^t} A \vec{x}$$

with A a symmetric $n \times n$ matrix with entries in R (this is why the 1/2 is there). This matrix is called the *hessian matrix.* Note that it has the second order partials. The *Gram matrix* of Q is $\frac{1}{2}A$ which is in $M_n(\frac{1}{2}R^n)$.

Example 2. $x^2 + xy + y^2$ can be written as

$$\frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Main Question: What values of R are represented by Q? i.e., what is $Q(R^n)$?

Notions of Equivalence The following are the different ways to view equivalence

- We consider quadratic forms $Q(\vec{x}) = \sum c_{ij} x_i x_j$ up to linear invertible changes of coordinates of R^n. We denote the equivalence by $Q_1 \sim_R Q_2$.
- Q_1 and Q_2 with gram matrices A_1 and A_2 are equivalent if and only if there is a $U \in GL_n(R)$ s.t. $U^t A_1 U = A_2$.

These two are equivalent because invertible linear changes of coordinates change the Gram matrix in precisely that way. e.g. If $\vec{x} = U\vec{y}$, then $\vec{x^t} A \vec{x} = Q(\vec{x}) = Q(U\vec{y}) = y^t \vec{U}^t A U \vec{y}$, so the Gram matrix for Q in $\vec{y}$ coordinates is precisely $U^t A U$.

The Determinant of a Quadratic Form We would like to define the determinant of Q as the determinant of its Hessian matrix, but up to equivalence this is not well defined. However, if Q_1 and Q_2 are equivalent, then their determiants differ by a square in R. Thus, we can define the determinant of Q as

$$\det(Q) = [\det(A)] \in R^*/(R^*)^2 \cup \{0\}.$$

Example 3. If $R = \mathbb{Z}$, then $\mathbb{Z}^* = \{\pm 1\}$ and so $(\mathbb{Z}^*)^2 = \{1\}$ and so the determinant of a quadratic form is well defined as an element of $\mathbb{Z}$.

Example 4. If $R = \mathbb{C}$, then $\mathbb{C}^* = \mathbb{C} - \{0\}$ and so $(\mathbb{C}^*)^2 = \mathbb{C}^*$ and so the only possibilities for the determinant of a quadratic form in this case are 0 or 1.

Direct Sums Let $Q_1$and Q_2 be forms in n and m variebles respectively. The direct sum $Q_1 \perp Q_2$ is to the from in $n + m$ variables defined by

$$Q_1 \perp Q_2(\vec{x}, \vec{y}) = Q_1(\vec{x}) + Q_2(\vec{y}).$$

Structure Theorem over fields

Definition 5. We say Q is *non-degenerate* if $\det(Q) \neq 0$.

Theorem 6. *If $R = k$ is a field, then $\det(Q) = 0$ implies that $Q \sim_k Q' \perp (0x^2)$*

Proof. Consider the associate bilinear form to Q: $B(\vec{x}, \vec{y}) = \vec{x}^t A \vec{y}$. Since $\det Q = 0$, then there is an $\vec{v} \neq \vec{0}$ s.t. $\vec{v}^t A = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}$. So, if we chose a basis with $\vec{v}$ as a first vector, then in this basis the matrix takes the form

$$\begin{bmatrix} 0 & * & \dots & * \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix},$$

but since the matrix is symmetric, then it actually looks like

$$\begin{bmatrix} 0 & \dots & 0 \\ \vdots & A' & \\ 0 & & \end{bmatrix}.$$

This A' defines Q' in the statement. □

By this theorem, we may assume that any quadratic fomr is non-degenerate, since we can always split it like above until we get something which is non-degenerate.

Equivalence via symmetric elementary operations We can understand the orbit of A under $U \cdot A = U^t A U$ for $U \in GL_n$ by decomposing U into elemetary operations (any matrix in GL_n is the product of elementary matrices). Elementary operations correspond to elementary colums operation on the right side, and elementary row operations on the left side in

$$\vec{x}^t A \vec{x}.$$

Example 7. Let $Q = x^2 + 2xy + y^2$ with associate Gram matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Change it by row operations, but every row operation needs to be acompanied by the equivalent column operation. If we rubtract the first row from the second, and then do the same with the comuns you get:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $Q \sim u^2$, where u is a linear function of x and y which was obvious from the beginning! $Q = (x+y)^2$.

Theorem 8. *Any quadratic form over a field k (char $k \neq 2$) can be diagonalized.*

Proof. Assume Q is non-degenerate and choose a basis. Let a be first element of the diagonal

$$\begin{bmatrix} a & * & \dots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{bmatrix}.$$

If $a \neq 0$, then we with row and column operations we can make all the other entries in the first row and column zero

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}.$$

If $a = 0$, then there is a nonzero entry in the first row (by non-degeneracy)

$$\begin{bmatrix} 0 & \dots & * & \dots \\ \vdots & & & \\ * & & * & \\ \vdots & & & \end{bmatrix}$$

and adding this column to the first column (and doing the corresponding row opperation), then we get a matrix with nonzero first element in the diagonal, and we can do what we did before in the case $a \neq 0$.

We have thus proved that we can find always modify a basis to one in which Q has the form

$$\left[\begin{array}{c|ccc} a & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right]$$

for some $a \neq 0$ and some non-degenerate matrix A. The process can be continued until one obtains a diagonal matrix with nonzero entries. □

Corollary 9. *Any quadratic form over $\mathbb{C}$ is equivalent to one of the form $x_1^2 + \ldots + x_n^2$, and any quadratic form over $\mathbb{R}$ is equivalent to one of the form $x_1^2 + \ldots + x_n^2 - (y_1^2 + \ldots + y_m^2)$.*

Proof. After diagonalizing one can rescale the basis elements and this can change the diaginal entries up to squares. See the example below. □

Example 10. Over $\mathbb{R}$ we have

$$4x^2 - 5y^2 \sim (2x)^2 - (\sqrt{5}y)^2 = u^2 - v^2,$$

while over $\mathbb{C}$ we have

$$4x^2 - 5y^2 \sim (2x)^2 + (i\sqrt{5}y)^2 = u^2 + v^2.$$

Over $\mathbb{Q}$ the best one can do is

$$4x^2 - 5y^2 \sim (2x)^2 - 5(y)^2 = u^2 - 5v^2.$$

Theorem 11. *If Q is non-degenerate, and $Q(\vec{v}) = 0$ with $\vec{v} \neq \vec{0}$, then*

$$Q \sim xy \perp Q'$$

Proof. If we take $\vec{v}$ as the first vector of a basis and let B be the asociated bilinear form. Since Q is non-degenerate then $B(\vec{v}, \cdot)$ is not identically zero and so, there is a $\vec{w}$ with$B(\vec{v}, \vec{w}) = a \neq 0$. Taking $\vec{w}/a$ as part of the second element of the basis we get a matrix for Q of the form

$$\left[\begin{array}{cc|c} 0 & 1 & \\ & & * \\ 1 & * & \\ \hline & & \\ & * & * \end{array}\right].$$

Moreover, by changing $\vec{w}/a$ to $b\vec{v} + \vec{w}/a$ for some b, we can make sure that the unknown entry in the 2 by 2 matrix on the top left is 0 (this will use the fact that $Q(\vec{v}) = 0$). By further scaling this new second vector we obtain a matrix of the form

$$\left[\begin{array}{cc|c} 0 & 1 & \\ & & * \\ 1 & 0 & \\ \hline & & \\ & * & * \end{array}\right].$$

Finally, using this form with row and column operations we can make sure that it looks in the form

$$\left[\begin{array}{cc|ccc} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \dots & * \end{array}\right].$$

□

Definition 12. We say Q is *isoptropic* if there is a $\vec{v} \neq \vec{0}$ with $Q(\vec{v}) = 0$. We say is is *anisotropic* if is is not isotropic.

Definition 13. We say Q is *universal* if $Q(R^n) = R$ (as sets).

Theorem 14. *If $R = k$ is a field, then H (the form $2xy$) is universal.*

Proof. Simple. □

Theorem 15. *If $R = \mathbb{R}$ then Q is universal iff it represents both positve and negative numbers.*

Proof. We may assume it is not degenerate, and so after diagonalizing it its matrix can be brought to the form

$$\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & -I_m \end{array}\right]$$

where I_n is the $n \times n$ identity matrix. Since the form represents both negative and positive numbers we know that $n, m \geq 1$. Thus Q has the subfrom $x^2 - y^2$ which is equivalent to xy and so represents all numbers. □