Parameter Spaces of Curves and Hypersurfaces

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Contents

1 Introduction

This term paper is based on two sets of exercises. The first set comes from our course text [Sha94], and grew into the contents of section 2. In it we discuss the parameter space of hypersurfaces and the hypersurface of singular hypersurfaces. The second set of exercises comes from Lecture 22 in Harris' book [Har92]. These form the backbone of section 3 which deals with the parameter space of conics in the plane.

My original idea was to develop the contents of section 4 regarding the parameter space of plane cubics as soon as I had finished with the exercises from Harris' book. However, it soon became clear that there was enough material in the solution of these exercises to make time run short. For this reason section 4 only contains a short discussion of the sort of things that could be expected, and some hints of how much more fun its hypothetical contents could be.

In this paper we will focus on the equations defining hypersurfaces and not on the varieties themselves. Thus, two different embeddings of the same hypersurface will be considered to be different objects, even if they differ by a change of coordinates (including those as trivial as multiplying a particular coordinate by a constant). Out of this will not obtain insight about the hypersurfaces themselves, but will encounter interesting geometric situations. This whole paper could be regarded as a very long (and fun) example of various concepts that arise in algebraic geometry, and various interesting applications of the theorems that we saw in class. I hope the reader enjoys its contents as much as I did while writing it.

2 The General Setup

Remember that in $k[x_0, \ldots, x_n]$ the set of homogeneous degree d polynomials is an $\binom{n+d}{d}$-dimensional vector space over k. We wish to study the family of projective hypersurfaces defined by these forms in $\mathbb{P}^n$, which we refer to by *degree d hypersurfaces*. Since constant multiples of a form define the same hypersurface, we may naturally identify the set of hypersurfaces of degree d in $\mathbb{P}^n$ with a projective space $\mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$. Explicitly, we identify the hypersurface with equation $\sum_I a_I x^I$ with the point $[a_I] \in \mathbb{P}^N$ (using multi-index notation), and we say that the space $\mathbb{P}^N$ is the *Parameter Space* of hypersurfaces of degree d in $\mathbb{P}^n$. Note that since a homogeneous degree d polynomial F may not be the generator of the ideal of $V(F)$, this space includes hypersurfaces of $\mathbb{P}^n$ which may be defined by forms of smaller degree.

These identifications allow one to study the family of hypersurfaces of a fixed degree geometrically, which is what we will be doing in this paper. In this first section we discuss some general theorems that can be proved in this general setting, and in the following sections we will specialize to some particular cases where we can give more information.

Before starting, however, we need to make a convention regarding terminology and the notion of singularity in particular. When studying the singularities of a hypersurface, one needs to deal with the fact that a homogeneous polynomial F may not be the generator of the ideal of $V(F)$. Since singularities are defined in terms of the generators of the ideal, working with the F directly will lead to erroneous conclusions. As an example, consider a linear form L and let $F = L^d$. If we use the criterion about all derivatives vanishing at a point to find the singularities of the hypersurface defined by F, we find that the whole of $V(F) = V(L)$ is singular. This of course is not the case, but we will define this to be so in this setting. Explicitly, we will say that *a hypersurface defined by F is singular at a point p if all partial derivatives of F vanish at p, regardless of whether F generates the ideal of $V(F)$ or not.* This convention will allow us to ignore this subtlety both in the statements and in the proofs.

For the remainder of this section n and d will be fixed. We can therefore refer to the N defined above without ambiguity and will do so henceforth. We will also denote by $[a_I]$ the coordinates of the parameter space $\mathbb{P}^N$.

2.1 Hypersurfaces through a fixed point

One of the simplest conditions one can impose on a hypersurface is to require that it pass through a specific point $p = [p_0 : \ldots : p_n]$. Let us see what this condition corresponds to in the parameter space $\mathbb{P}^N$.

If I is any degree d multi-index, and we denote by p^I the result of replacing x_i by p_i in the monomial x^I, the condition that the hypersurface $\sum_I a_I x^I = 0$ contain the point p translates to $\sum_I a_I p^I = 0$. This last condition is linear on the coefficients a_I and so defines a hyperplane in $\mathbb{P}^N$. This gives us our first result:

2.1.1 Proposition. *The set of degree d hypersurfaces containing a particular point is a hyperplane in $\mathbb{P}^N$.*

This apparently simple result already gives a very nice result about hypersurfaces containing a certain number of specified points.

2.1.2 Corollary. *There is a hypersurface of degree d in $\mathbb{P}^n$ passing through N arbitrary points of $\mathbb{P}^n$.*

Proof. The condition that each point be contained in the hypersurface gives a hyperplane in $\mathbb{P}^N$, and the intersection of N hyperplanes in $\mathbb{P}^N$ is non-empty. □

Note that one cannot guarantee the existence of a hypersurface through more than N points since the hyperplanes may not have a common intersection.

In the other direction, the independence of the conditions that a hypersurface pass through a given set of points is a delicate one, which is the reason why we can't guarantee uniqueness of the hypersurface in the previous corollary. Even if the N points are chosen to be linearly independent, the equations imposed on the coefficients will not be independent in general since they are not linear in the coordinates of the point. A good example showing the subtleties if this issue is the following:

2.1.3 Example. Consider conics in the plane where $d = 2$ and $N = 5$. Fix a line L and let A, B be distinct points on L. Then the conditions that a conic contain A and B are obviously independent since not every conic containing A contains B. If we take another point $C \in L$, then the extra condition that the conic contain C is independent of the other two since it forces the conic to contain L by Bezout's theorem. If $D \in L$ is another point, then the condition that the conic pass through D is not independent of the other three, since it is automatically satisfied.

The previous example shows why statements like "five points determine a conic in $\mathbb{P}^2$", or "nine points determine a cubic in $\mathbb{P}^2$" are false unless one imposes some condition on the arrangement on the points. Uniqueness depends on the configuration of the points.

2.2 The locus of reducible hypersurfaces

2.2.1 Proposition. *The locus Γ of points in $\mathbb{P}^N$ corresponding to hypersurfaces defined by reducible polynomials is a closed set.*

Remark. Note that this theorem is not referring directly to the irreducibility of the hypersurface but to the defining polynomial. It hay happen that the form is reducible while the hypersurface is irreducible, but this will only be the case when the form is a power of an irreducible form of lower degree.

Proof. For $k = 1, \ldots, d-1$ let Γ_k be the locus corresponding to the hypersurfaces which split up as a union of a hypersurface of degree k and a hypersurface of degree $d-k$ (corresponding to a factorization of the defining polynomial into factors of these degrees). Then $\Gamma = \bigcup \Gamma_k$ and it is enough to prove statement for a fixed Γ_k. Now, consider the parameter spaces $\mathbb{P}^{N_k}$ and $\mathbb{P}^{N_{d-k}}$ of hypersurfaces of degree k and $d-k$ respectively. Then we see that Γ_k is the image of the map $\phi_k : \mathbb{P}^{N_k} \times \mathbb{P}^{N_{d-k}} \to \mathbb{P}^N$ sending the coefficients of the polynomials to the coefficients of their product. If we show this map is a regular map between projective varieties then we will obtain that Γ_k is closed since it will be the image of a projective variety under a regular map. The components of this map are obviously polynomials in the entries, so to verify that ϕ is a regular map between projective varieties we need to show that we can't have $\phi([b_J], [c_K]) = [0]$ for any $([b_J], [c_K]) \in \mathbb{P}^{N_k} \times \mathbb{P}^{N_{d-k}}$. However, this is clear since one of the b_J and one of the c_K is non-zero so we have two non-zero polynomials in $k[x_0, \ldots, x_n]$ and their product is a non-zero polynomial. $\square$

Note that in the proof above $\Gamma = \bigcup \Gamma_k$ is the decomposition of Γ into its irreducible components since each Γ_k is irreducible being the image of an irreducible variety (note however that $\Gamma_k = \Gamma_{d-k}$).

It would be interesting to find the dimensions of the Γ_k. In that direction, note that Γ_k is of dimension less than (or equal to) $N_k + N_{d-k} = \binom{n+k}{k} + \binom{n+d-k}{d-k} - 2$. A way to compute the dimension of Γ_k would be to study the dimensions of the fibers of ϕ_k. Note also that the ϕ_k are not one to one.

2.3 The locus of singular hypersurfaces

Since any reducible hypersurface is singular, the locus Γ of hypersurfaces defined by reducible polynomials is a subset of the locus Σ of singular hypersurfaces in the parameter space $\mathbb{P}^N$ (Σ standing for Σingular). In this section we prove that Σ is also a closed subset of $\mathbb{P}^N$, thus showing that Γ is a projective subvariety of the projective variety Σ. To prove the result we need a preliminary result which is interesting in its own right.

2.3.1 Proposition. *Let $F_0, \ldots, F_n \in k[x_0, \ldots, x_n]$ be homogeneous polynomials of degrees $m_0, \ldots, m_n$ respectively. There exists a polynomial in the coefficients of the F_i which is homogeneous in the coefficients of each F_i, and which is zero if and only if $V(F_0, \ldots, F_n) \neq \emptyset$ in $\mathbb{P}^n$.*

Remark. If we had n forms instead of $n+1$ then we would know that the variety defined by them is non-empty by the general theorems about intersections of hypersurfaces in $\mathbb{P}^n$. This proposition is telling us that for the first interesting case, the condition that there be a non-empty intersection of the hypersurfaces is an algebraic condition in the coefficients.

Proof. Let $\mathbb{P}^{N_i}$ be the space of forms of degree m_i for each i and define the set $X \subset \mathbb{P}^n \times \prod_{i=0}^n \mathbb{P}^{N_i}$ by

$$X = \{[x : F_0 : \ldots : F_n] \mid x \in \mathbb{P}^n,\ F_i \text{ form of degree } m_i \text{ and } F_i(x) = 0\},$$

where we are identifying the form F_i with its coefficients as usual. Then X is a projective variety in $\mathbb{P}^n \times \prod_{i=0}^n \mathbb{P}^{N_i}$ since it is defined precisely by $F_0 = \ldots = F_n = 0$ where we view the coefficients of the F_i as coordinates.

Consider the projection $\psi : X \to \mathbb{P}^n$ given by

$$[x : F_0 : \ldots : F_n] \mapsto x$$

which is clearly surjective since any $x \in \mathbb{P}^n$ is the zero of at least one form of degree m_i for all i. If $p \in \mathbb{P}^n$, then the condition that the form F_i vanish at p defines a hyperplane in $\mathbb{P}^{N_i}$ as was explained in 2.1.1, and this implies that the fiber of ψ above p is a linear subspace of dimension $\sum(N_i - 1) = (\sum N_i) - (n+1)$ of $\mathbb{P}^n \times \prod_{i=0}^n \mathbb{P}^{N_i}$. Therefore, the fibers of ψ all have the same dimension and so it follows both that X is irreducible and that[1]

$$\begin{aligned} \dim X &= \dim \psi(X) + \dim(\text{fibers of } X) \\ &= n + \sum N_i - (n+1) \\ &= \left(\sum N_i\right) - 1. \end{aligned}$$

Consider now the projection $\phi : X \to \prod_{i=0}^n \mathbb{P}^{N_i}$ given by

$$[x : F_0 : \ldots : F_n] \mapsto [F_0 : \ldots : F_n].$$

Note that the image $\phi(X)$ of ϕ corresponds precisely to $n+1$-tuples of forms of degrees $m_0, \ldots, m_n$ which have a common zero in $\mathbb{P}^n$. To finish the proof we need to show that $\phi(X)$ is a hypersurface in $\prod_{i=0}^n \mathbb{P}^{N_i}$. We have that $\phi(X)$ is both closed and irreducible since ϕ is a morphism and X is both irreducible and projective. Moreover, by the computation of the dimension of X given above we have $\dim \phi(X) \le (\sum N_i) - 1$. We will prove that actually $\dim \phi(X) = (\sum N_i) - 1$ which will imply that $\phi(X)$ is a codimension 1 irreducible variety of $\prod_{i=0}^n \mathbb{P}^{N_i}$, and thus a hypersurface as desired. To prove the equality of the dimensions it is sufficient to exhibit a fiber of dimension zero by the theorem on the dimension of the fibers (see footnote). This fiber is easy to construct, just take $F_0 = x_0^{m_0}$, $F_1 = x_1^{m_1}, \ldots, F_{n-1} = x_{n-1}^{m_{n-1}}$ and for F_n take $F_n = x_0^{m_n}$. Then $V(F_0, \ldots, F_n) = \{[0 : \ldots : 0 : 1]\}$ and so the fiber above $[F_0 : \ldots : F_n]$ only consists of one point. □

Remark. Note that the proof of the above proposition is non-constructive since we have proved that an equation exists, but have not given a way to find it. Even though particular cases have been studied extensively, there is no known explicit description for this polynomial. The interested reader may take a look at [Stu97] for a survey of (classical) elimination theory where this issue is discussed. For now we note that we already know a particular case of this result as the following example shows. We will also discuss this a little further in section 4.

2.3.2 Example. If the forms $F_0, \ldots, F_n$ are all linear, say $F_i = \sum_j a_{ij} x_j$, then $V(F_0, \ldots, F_n) \neq \emptyset$ if and only if $\det(a_{ij}) = 0$.

We can now give the application of this result to singular degree d hypersurfaces.

2.3.3 Proposition. *Assume that char(K) does not divide d. The locus $\Sigma \subset \mathbb{P}^N$ corresponding to the singular hypersurfaces is closed. Moreover, Σ is itself a hypersurface in $\mathbb{P}^N$.*

[1] See Shafarevich [Sha94] Chapter *I*, Section 6, Theorems 8 and 7 (in this order).

Proof. A hypersurface defined by $F = \sum_I a_I x^I$ is singular if and only if the system of equations $\partial F/\partial x_0 = \ldots = \partial F/\partial x_n = 0$ has a solution in $\mathbb{P}^n$. By the previous proposition applied to $m_i = d-1$, there is a polynomial in the coefficients of forms of degree $d-1$ which is zero if and only if the forms have a common zero. Say this polynomial is $R([F_0],[F_1],\ldots,[F_n])$, where by $[F_i]$ we mean the tuple with the coefficients of the forms. Then $R([\partial F/\partial x_0],\ldots,[\partial F/\partial x_n])$ is a homogeneous polynomial in the coefficients of F, that is, in the $[a_I]$, which is zero if and only if the hypersurface defined by F is non-singular. This proves that Σ is a hypersurface in $\mathbb{P}^N$. □

2.3.4 Example (Quadrics)**.** Assume $char(K) \neq 2$. A form of degree two $F = \sum_{i\leq j} a_{ij}x_ix_j$ defines a quadric in $\mathbb{P}^n$. Then $\partial F/\partial x_k$ is a linear form with coefficients among the a_{ij} for all k (the a_{ii} get multiplied by 2) and so, by the example above, $V(F)$ is singular if and only if the determinant of a matrix involving the a_{ij} vanishes. Explicitly, the hypersurface of singular quadrics is defined by

$$\det \begin{bmatrix} 2a_{00} & a_{01} & & a_{0n} \\ a_{01} & 2a_{11} & & \vdots \\ \vdots & & \ddots & \\ a_{0n} & \ldots & & 2a_{nn} \end{bmatrix} = 0.$$

3 The Space of Plane Conics

We now study the particular case $n = 2$, $d = 2$ of conics in $\mathbb{P}^2$ in more detail. Throughout this section assume that char $k \neq 2$. We will write the general equation of a conic as

$$a_{00}x^2 + a_{11}y^2 + a_{22}z^2 + a_{01}xy + a_{02}xz + a_{12}yz = 0.$$

Note that this determines both our choice for the coordinates $[x : y : z]$ of $\mathbb{P}^2$ and the coordinates $[a_{00} : a_{11} : a_{22} : a_{01} : a_{02} : a_{12}]$ for the parameter space $\mathbb{P}^5$ of conics.

3.1 The varieties of double lines and singular conics

If the form F defining a conic is reducible, then F is either the product of two distinct linear forms or the square of one linear form. When F is a square we say that the conic is a *double line.* We will denote by Δ the locus in $\mathbb{P}^5$ corresponding to double lines (Δ standing for Δouble), and following the notation from the section 1, we will denote the locus of conics defined by reducible F by Γ, and the locus of singular conics by Σ. Remember also our convention about singularities which implies that any conic in Δ is singular (as a conic) even though as an algebraic set it is a smooth irreducible line.

Note then that we have the inclusions

$$\Delta \subset \Gamma \subseteq \Sigma \subset \mathbb{P}^5.$$

In section 1 we proved that both Γ and Σ are closed. Explicitly, Σ is the 4-dimensional hypersurface defined by

$$\det \begin{bmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix} = 0,$$

and Γ is the image of the morphism $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ sending the coefficients of two linear forms to the coefficients of the conic defined by their product. Note also that Δ is the image of the diagonal in the above map, which shows it is also closed. We can actually say much more about Δ:

3.1.1 Proposition. *Δ is closed, irreducible, two-dimensional, and trivially isomorphic to the image of the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$.*

Proof. Closedness we proved above, and irreducibility follows since Δ is the image of the irreducible diagonal. Moreover, note that Δ is also the image of the map sending the line defined by $l = ax + by + cz$ to the "conic" l^2. In coordinates this is given by

$$[a : b : c] \mapsto [a^2 : b^2 : c^2 : 2ab : 2ac : 2bc]$$

which we recognize as a disguised version of the Veronese map $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 : [a : b : c] \mapsto [a^2 : b^2 : c^2 : ab : ac : bc]$. Specifically, composing the above map with the change of coordinates $[a_{00} : a_{11} : a_{22} : a_{01} : a_{02} : a_{12}] \mapsto [a_{00} : a_{11} : a_{22} : a_{01}/2 : a_{02}/2 : a_{12}/2]$ we get ν_2. □

The relationship between Δ and the Veronese surface in $\mathbb{P}^5$ allows us to give explicit equations for Δ.

3.1.2 Corollary. *Δ is defined by the vanishing of the two by two minors on the matrix*

$$\begin{bmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix}.$$

Proof. The equations defining the image of the Veronese map can conveniently be expressed as the two by two minors of the symmetric matrix (see [Har92], example 2.6)

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}.$$

The statement follows by making a change of variables taking this surface to Δ, which in terms of the matrices takes the matrix above to the matrix in the statement multiplied by 1/2. □

Regarding Γ, we have:

3.1.3 Proposition. Γ *is* 4*-dimensional and irreducible.*

Proof. Remember that Γ is the image of the morphism $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ sending the coefficients of two linear forms to the coefficients of the conic defined by their product. This implies that Γ is irreducible since $\mathbb{P}^2 \times \mathbb{P}^2$ is. For the statement about the dimension, note that the fibers of the above map are of size at most two since by unique factorization the only thing that we can change is the order of the factors. This implies that $\dim \Gamma = \dim \mathbb{P}^2 \times \mathbb{P}^2 = 4$. □

3.1.4 Corollary. $\Sigma = \Gamma$ *(i.e., a conic is reducible if and only if it is singular).*

Proof. Say f is the the determinant polynomial defining Σ. Considering $f \in k[a_{11}, a_{22}, \ldots, a_{12}][a_{00}]$, we can write $f = \alpha a_{00} + \beta$ with $\alpha, \beta \in k[a_{11}, a_{22}, \ldots, a_{12}]$. Any factorization of f would necessarily involve a factorization of both α and β in $k[a_{11}, a_{22}, \ldots, a_{12}]$ since a_{00} can't appear on both factors. However, $\alpha = a_{11}a_{22} - a_{12}^2$ is irreducible so this cannot happen. This implies that f itself is irreducible and so we get that Σ is irreducible.

The result follows from the fact that $\Gamma \subseteq \Sigma$ and both are irreducible, closed, and 4-dimensional. □

Remark. Note that the corollary does not hold for quadrics in higher dimensions as one can see with $x^2 + y^2 = z^2$ in $\mathbb{A}^3$.

We state everything we have proved up until now in a theorem for future reference.

3.1.5 Theorem. *A conic is singular if and only if it is irreducible. The loci* Δ *of double lines and* Σ *of singular conics (or equivalently of unions of lines) are both closed and irreducible.* Σ *is a* 4*-dimensional hypersurface defined the vanishing of the determinant of*

$$\begin{bmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix},$$

and $\Delta \subset \Sigma$ *is* 2*-dimensional and defined by the vanishing of the two by two minors of this same matrix.*

3.2 Quadratic forms and the action on $\mathbb{P}GL_3$ on the space of conics

In the previous section all the tools we used were essentially geometric. In this section we re-derive the equations defining Σ and Δ as well as the proof that $\Gamma = \Sigma$ using the theory of quadratic forms. The usefulness of this approach lies in the fact that in this setting the formulas we obtained above are more motivated. However, statements regarding the dimension or irreducibility of Δ are not direct. We will also take a closer look at the induced action of $\mathbb{P}GL_3$ on the space of conics. Understanding this action will be of considerable importance in what follows.

Remember that to any homogeneous degree two polynomial

$$F = a_{00}x^2 + a_{11}y^2 + a_{22}z^2 + a_{01}xy + a_{02}xz + a_{12}yz,$$

we can associate a quadratic form given by $v \mapsto v^t A v$, where $v = [x : y : z]$ and the matrix A is the symmetric matrix given by

$$\begin{bmatrix} a_{00} & \frac{1}{2}a_{01} & \frac{1}{2}a_{02} \\ \frac{1}{2}a_{01} & a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{02} & \frac{1}{2}a_{12} & a_{22} \end{bmatrix},$$

which we may recognize as the matrix from 3.1.5 multiplied by 1/2. Now, since we are identifying the $F's$ up to multiplication by a constant, we identify these matrices up to multiplication by a constant. This gives us two ways of thinking about the space of conics: either as the space $\mathbb{P}^5$ or as the projective space of 3×3 symmetric matrices with the identification explained above.

Now, $\mathbb{P}GL_3$ acts on $\mathbb{P}^2$ by changes of coordinates, and this induces an action on the space of conics by taking the conic (in any of its incarnations) to the resulting conic after the change of coordinates. If $T \in \mathbb{P}GL_3$ is the change of coordinates, then in terms of matrices the induced transformation is given by $\Phi_T : A \mapsto (T^{-1})^t A T^{-1}$ since if the equation of the conic is $v^t A v = 0$ and $u = Tv$ are the new coordinates, then the resulting equation is $(T^{-1}u)^t A(T^{-1}u) = 0$.

Note that Φ_T is linear in the entries of A, and it has an inverse $A \mapsto T^t A T$ which is induced by the change of coordinates given by T^{-1}. Therefore, in terms of $\mathbb{P}^5$, Φ_T is a linear automorphism and so in fact a linear change of coordinates of $\mathbb{P}^5$ given by some 6×6 matrix $\in \mathbb{P}GL_6$ whose entries depend non-linearly in the entries of T.

The following observation will be of great use. It states that the induced action of $\mathbb{P}GL_3$ on the space of conics is compatible with the identifications we are making. This is obvious because we constructed it to be this way.

3.2.1 Observation. *Let $C \subset \mathbb{P}^2$ a fixed conic and fix $T \in \mathbb{P}GL_3$. Let A_C and $p_C \in \mathbb{P}^5$ be matrix and point corresponding to C, and let $A_{T(C)}$ and p_{TC} be the matrix and point corresponding to $T(C)$. Then in terms of matrices $\Phi_T(A_C) = A_{T(C)}$ and in terms of $\mathbb{P}^5$, $\Phi_T(p_C) = p_{T(C)}$.*

In the theory of quadratic forms it is proved that one can always make a linear change of variables of $\mathbb{P}^2$ so that the matrix $A_{T(C)}$ becomes diagonal with ones or zeros on the diagonal [2]. This implies that any conic is by a change of variables isomorphic to one of the three conics $x^2 + y^2 + z^2 = 0$, $x^2 + y^2 = 0$ (the union of two lines) or $x^2 = 0$ (a double line). Which one of these it is is determined by the rank of the matrix A_C, which is unchanged by transformations of the form $A \mapsto D^t A D$ with $D \in \mathbb{P}GL_3$. We therefore see that in terms of matrices Σ is the set of rank ≤ 2 matrices, and Δ is the set of rank 1 matrices.

Note that this implies that the only non-singular conics are the ones of rank 3, and so we obtain directly that $\Gamma = \Sigma$ (remember that lines are singular as conics). We also obtain the equations defining Δ and Γ since Γ is defined by the condition that the associated matrix have rank ≤ 2, which we may write as

$$\det \begin{bmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix} = 0,$$

and Δ is defined by the condition that the matrix have rank 1, which is equivalent to the vanishing of all the two by two minors. We have therefore obtained the results of Theorem 3.1.5 by different means.

The above discussion also proves the following proposition which we will use a considerable number of times in what follows.

3.2.2 Proposition. *Let $T \in \mathbb{P}GL_3$ be a change of coordinates. Then Φ_T restricts to automorphisms of both Σ and Δ. In particular, $\Phi_T(\Sigma) = \Sigma$ and $\Phi_T(\Delta) = \Delta$.*

Remark. Note that if we exclude the degenerate case of double lines, the argument given above that $\Sigma = \Gamma$ also proves that a conic is reducible if and only if it is singular where reducible and singular now have their standard meaning, and not the special one we are giving them for the parameter space of curves of degree 2.

3.3 Conics containing fixed points or lines

We return now to subject that was discussed in 2.1 regarding conditions on the coefficients imposed by points. For the sake of being explicit, we note that in this case if the coordinates of $p \in \mathbb{P}^2$ are $[p_0 : p_1 : p_2]$, then the linear subspace of $\mathbb{P}^5$ of conics containing p is the hyperplane defined by

$$p_0^2 a_{00} + p_1^2 a_{11} + p_2^2 a_{22} + p_0 p_1 a_{01} + p_0 p_2 a_{02} + p_1 p_2 a_{02} = 0.$$

In example 2.1.3 we discussed briefly a particular choice of points and the independence of the corresponding equations (also referred to as conditions imposed by the points). Note also that proposition 2.1.2 implies that

[2] See [Har92] example 3.3 for the coordinate free version of this result, or [Kna06] Chapter VI, Theorem 6.5 for the matrix version.

there is always a conic passing through any five points in $\mathbb{P}^2$. The following proposition gives a condition which guarantees that the conditions imposed by each point are independent giving uniqueness.

3.3.1 Proposition. *There is a unique conic passing through five specified points if no four of them are collinear.*

Proof. Let $p_1, \ldots, p_5$ be five points with no four of them collinear and let C, C' be conics that contain all the p_i. Then C and C' intersect at 5 points and so by Bezout's theorem C and C' must contain a common component. If they are irreducible then $C = C'$ so we assume this is not the case. Therefore C and C' are both reducible and contain a common line. However, if two lines contain the five points $p_1, \ldots, p_5$ and no four of them are collinear, then one line must contain exactly three, and the other the remaining two. This determines C uniquely and so $C = C'$. $\square$

Regarding the conditions that a conic contain a fixed line remember that Bezout implies that if a conic contains three collinear points then it contains the line through them (so in particular is reducible and singular). This will be of use in the following proposition.

3.3.2 Proposition. *Let $l \subset \mathbb{P}^2$ be a line. The locus $\Sigma_l \subset \mathbb{P}^5$ of conics that contain l is an irreducible 2-dimensional linear subspace of $\mathbb{P}^5$.*

Proof. Let $[l_0 : l_1 : l_2]$ be the coordinates of l in the dual of $\mathbb{P}^2$ (which we identify with $\mathbb{P}^2$ as usual), meaning that l is defined by $l_0x + l_1y + l_2z = 0$. Then Σ_l is the image of the set $\{[l_0 : l_1 : l_2]\} \times \mathbb{P}^2$ under the morphism $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ sending two lines to the conic defined by their product. This proves irreducibility and the statement regarding the dimension.

To prove that Σ_l is a linear subspace one can look more closely at the map above, or alternatively use the following argument which gives us as a bonus a set of explicit equations defining Σ_l. As explained above, a conic contains l if and only if it contains three arbitrary (but fixed) points $p_1, p_2, p_3 \in l$. The conics that contain p_i are a hyperplane H_i in $\mathbb{P}^5$ and we have $\Sigma_l = H_1 \cap H_2 \cap H_3$ which proves that Σ_l is a linear subspace, and moreover gives us explicit equations defining Σ_l. $\square$

Remark. Note that of course $\Sigma_l \subset \Sigma$. In some sense Σ is ruled by these $\Sigma_l \cong \mathbb{P}^2$ for varying l.

3.4 Singularities of Σ and Δ

Since Σ is an irreducible hypersurface, its singular locus is the set points defined by the vanishing of the partial derivatives of its defining equation.

This gives us the six equations which define the singular locus of Σ:

$$\begin{aligned}
4a_{11}a_{22} - a_{12}^2 &= 0 \\
4a_{00}a_{22} - a_{02}^2 &= 0 \\
4a_{00}a_{11} - a_{01}^2 &= 0 \\
a_{02}a_{12} - 2a_{01}a_{22} &= 0 \\
a_{01}a_{12} - 2a_{02}a_{11} &= 0 \\
a_{01}a_{02} - 2a_{12}a_{00} &= 0,
\end{aligned}$$

which one may as the equations for the vanishing of all the two by two minors of the matrix of theorem 3.1.5. We thus obtain the following surprising result.

3.4.1 Proposition. *The singular locus of* Σ *is precisely* Δ*.*

Remark. Note that this result does not follow from the fact that the derivative of the determinant function with respect to a given entry is the minor corresponding to that entry since the matrix from theorem 3.1.5 is symmetric and each variable appears in more than one entry in the matrix.

A more direct proof for the above result (that is, one that does not rely on writing the explicit equations for the singular locus an noting that they define Δ) can be found in example 3, Chapter II, section 1.4 from Shafarevich's text [Sha94]. Following the example we identify $\mathbb{P}^5$ with the projective space of symmetric 3×3 matrices and in the example we restrict the dimensions and ranks to our particular case. The arguments in the text then prove that if A is a non-zero singular symmetric matrix then

$$\frac{d}{dt}\det(A + tB)|_{t=0} = 0$$

for all symmetric matrices B if and only if the rank of A is 1. This implies that the singular locus is precisely the rank 1 matrices, that is, it is precisely Δ. (Note that the notation in Shafarevich's example and our notation do not agree: what he refers to by Δ is our Σ).

Regarding Δ we have:

3.4.2 Proposition. Δ *is smooth.*

Proof. Note that Δ is isomorphic to the Veronese surface in $\mathbb{P}^5$ which in turn is isomorphic to $\mathbb{P}^2$. Smoothness follows. □

3.5 Tangent planes to Σ and Δ

In this section we determine the (embedded projective) tangent spaces to Σ and Δ at their smooth points. We first fix some notation.

Given a line L defined by a linear form $l_0x + l_1y + l_2z$, we denote by L both the line itself as a subset of $\mathbb{P}^2$ and the point $[l_0 : l_1 : l_2]$ in the dual space corresponding to L. For any two lines L and M we denote by $L \cdot M$ both the conic defined by the product of their defining forms, and the point in $\mathbb{P}^5$ to which this conic corresponds to. The correct meaning will hopefully be quite clear from the context.

3.5.1 Proposition. *Let L, M be distinct lines. The (embedded projective) tangent space to Σ at $L \cdot M$ is the space of conics containing the point $p = L \cap M$.*

Proof. First assume that $L \cdot M$ is the conic $xy = 0$, that is, that $L \cdot M = [0 : 0 : 0 : 1 : 0 : 0]$. Remember now that the tangent space at a point p on a general projective hypersurface in $\mathbb{P}^n$ defined by $F(x_0, \ldots, x_n)$ is given by the hyperplane $\sum_i x_i \partial F/\partial x_i(p) = 0$. In our particular case F is given by the determinant of the matrix from 3.1.5, and the partial derivatives are explicitly given right before 3.4.1. Using these equations we find that the equation of the tangent space of Σ at $L \cdot M$ is given by $a_{22} = 0$.

Now note that the space $a_{22} = 0$ can be described invariantly: A conic corresponds to a point with $a_{22} = 0$ if and only if the point $[0 : 0 : 1]$ is contained in the conic. Moreover, the point $[0 : 0 : 1]$ is precisely the point of intersection of the lines defined by $x = 0$ and $y = 0$. We have thus proved the result in the special case when the conic is defined by $xy = 0$.

For arbitrary L and M consider a change coordinates in $\mathbb{P}^2$ by an element $T \in \mathbb{P}GL_3$ taking L to the line $x = 0$ and M to $y = 0$. This is possible since $\mathbb{P}GL_3$ acts 3-transitively on $\mathbb{P}^2$, so we can send $p = L \cap M$ to $[0 : 0 : 1]$ and some other arbitrary points on L and M to $[1 : 0 : 0]$ and $[0 : 1 : 0]$ respectively. Consider now the change of coordinates T^{-1} and denote by Φ the induced change of coordinates $\Phi_{T^{-1}}$ of $\mathbb{P}^5$ as explained in 3.2.

In what follows we will use the contents of 3.2 substantially. Proposition 3.2.2 and observation 3.2.1 will be of particular importance.

Note that since $\Phi(\Sigma) = \Sigma$ then Φ takes the (projective embedded) tangent space of Σ at a point Q to the tangent space to Σ at $\Phi(Q)$. In particular, Φ takes the tangent space of Σ at xy to the tangent space of Σ at $L \cdot M$ since $T^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ takes the conic $xy = 0$ to $L \cdot M$. Now, as proved above, a conic is in the tangent space to Σ at xy if and only if the conic contains $[0 : 0 : 1]$, and under the change of coordinates T^{-1} this conic is taken to a conic that contains $p = L \cap M$ since T^{-1} takes $[0 : 0 : 1]$ to p. Thus, the tangent space of Σ at $L \cdot M$ is contained in the space of conics containing p and is therefore the whole space of conics containing p by dimensional and irreducibility arguments. □

There is also a very direct proof of the above result which does not use changes of coordinates. Even though it is not very enlightening, we include it below because it gives us the explicit equation for the tangent plane in terms of the coefficients of the defining equations of the lines. For this we need a preliminary lemma which gives coordinates of the intersection point of two lines in terms of the coefficients of their equations.

3.5.2 Lemma. *Let L, M be distinct lines in $\mathbb{P}^2$ defined by $l_0x+l_1y+l_2z=0$ and $m_0x+m_1y+m_2z=0$ respectively. The projective coordinates of the point p of intersection of the two lines is given by the cross-product of the vectors (l_0,l_1,l_2) and (m_0,m_1,m_2) in k^3 .*

Proof. This is just analytic geometry. A point $[x_0 : y_0 : z_0]$ is contained in L if and only if the vectors (l_0,l_1,l_2) and (x_0,y_0,z_0) are perpendicular in k^3. The set of vectors perpendicular to both (l_0,l_1,l_2) and (m_0,m_1,m_2) in k^3 is a one dimensional linear space generated by the cross product of the vectors. The result follows. □

Direct computational proof of 3.5.1. We remind again that the tangent space at a point p on a general projective hypersurface in $\mathbb{P}^n$ defined by $F(x_0,\dots,x_n)$ is given by the hyperplane $\sum_i x_i\partial F/\partial x_i(p)=0$. In our particular case F is given by the determinant of the matrix from 3.1.5, and the partial derivatives are explicitly given right before 3.4.1. Denoting L and M by $[l_0 : l_1 : l_2]$ and $[m_0 : m_1 : m_2]$ respectively, we obtain that

$$L\cdot M=[m_0l_0 : l_1m_1 : l_2m_2 : l_0m_1+l_1m_0 : l_0m_2+l_2m_0 : l_1m_2+l_2m_2].$$

Using these equations, one may explicitly write the equation of the tangent space to Σ at $L\cdot M$ by evaluating the partial derivatives at the point $L\cdot M$ (we omit the explicit equations here).

In another direction, using the lemma above, one may write down the equation of the hyperplane of conics containing the point $p=L\cap M$ in terms of the m_i and the l_i.

What we get from these computations is that these two hyperplanes are the same. This proves the proposition. □

We now come to the analysis of the tangent spaces to Δ:

3.5.3 Proposition. *The tangent space of Δ at L^2 is the space Σ_L of conics containing L.*

Proof. We work with the explicit equations defining Δ. Since Δ is not a hypersurface, in the following arguments we need to use the fact that our equations for Δ generate the ideal of Δ and refer the reader to [Har92] exercise 5.4 for the statement of the relevant exercise.

We first compute the tangent space at the conic given by $x^2=0$, that is, at the point $p=[1:0:0:0:0:0]$. For this we take a general point $b=[b_{00} : b_{11} : b_{22} : b_{01} : b_{02} : b_{12}]\in\mathbb{P}^5$, and evaluate each of the defining equations of Δ at $p+tb$. The tangent space consist of those b for which all these equations are divisible by t^2. This is equivalent to requiring all the minors of the matrix

$$\begin{bmatrix} 2+2tb_{00} & tb_{01} & tb_{02} \\ tb_{01} & 2tb_{11} & tb_{12} \\ tb_{02} & tb_{12} & 2tb_{22} \end{bmatrix}$$

to be divisible by t^2. Thus, b is in the tangent space if and only if $b_{11} = b_{22} = b_{12} = 0$. This space can be described invariantly: it is precisely the space of conics containing the line $x = 0$ since we are requiring the coefficients of the terms not involving x to be zero. This proves the proposition in this particular case.

An argument similar to the one given in the proof of 3.5.1 then proves that the tangent space to Δ at an arbitrary double line L^2 is the space of conics containing L by using 3.2.1 and 3.2.2 and an appropriate change of coordinates. □

Remark. Note that in 3.3.2 we had already mentioned the space Σ_L, and had proved that it was linear, two-dimensional, irreducible, and had given explicit equations defining it.

The proposition immediately implies the following surprising fact:

3.5.4 Corollary. *The union of all the (embedded) tangent spaces to Δ is precisely Σ. In other words, Σ is the tangent variety of Δ.*

3.6 The tangent cone to Σ at points of Δ

It is natural to wonder just how singular Σ is along Δ. In this section we compute the tangent cone of Σ at the points of Δ to answer this question.

We first compute the tangent cone at the point $p = [1 : 0 : 0 : 0 : 0 : 0]$ corresponding to the double line $x^2 = 0$. Note that in the affine patch corresponding to $a_{00} \neq 0$ this point conveniently corresponds to the origin $(0, 0, 0, 0, 0)$. In what follows we will use the same $(a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$ as affine coordinates by abuse of notation.

Remember that the equation of the tangent cone at the origin of an affine hypersurface is defined to be the zero locus of the homogeneous part of lowest degree of the polynomial generating its ideal. In our case the hypersurface is defined by

$$\det \begin{bmatrix} 2 & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix} = 0,$$

and the homogeneous part of lowest degree is given by $2(4a_{11}a_{22} - a_{12}^2)$. Therefore, the affine tangent cone of Σ at p is the hypersurface defined by $4a_{11}a_{22} - a_{12}^2 = 0$. Taking the projective closure we obtain the projective embedded tangent cone, given by the same equation. By the degree of this equation we also see that the multiplicity of Σ at p is 2.

We now proceed to give an invariant description of this hypersurface to obtain a general result.

3.6.1 Lemma. *The projective hypersurface H defined by $4a_{11}a_{22} - a_{12}^2 = 0$ consists of the double lines together with the conics tangent to or containing the line $x = 0$.*

Proof. H obviously contains the double lines since the equation defining H is one of the equations defining Δ (that is, it is one of the minors of the matrix of theorem 3.1.5). So assume now that $a = [a_{00} : \ldots : a_{12}] \in H - \Delta$. Now, $x = 0$ is tangent to or is contained in the conic corresponding to a if and only if the equation $a_{11}y^2 + a_{22}z^2 + a_{12}yz = 0$, resulting from setting $x = 0$ in the equation of the conic has a double (projective) root $[y_0 : z_0]$. The well know result regarding equations of degree two in one variable tells us that this is the case if and only if the discriminant vanishes. That is, if and only if $4a_{11}a_{22} - a_{12}^2 = 0$. This is precisely the equation defining H. □

Remark. Note that Δ has to be excluded in the statement "$x = 0$ is tangent to or is contained in the conic corresponding to a if and only if the equation $a_{11}y^2 + a_{22}z^2 + a_{12}yz = 0$ has a double root" because any double line intersects $x = 0$ with a double root (as long as it is not x^2 itself).

Note now that the equation defining H is a quadratic form in the variables $a_{00}, \ldots, a_{12}$ of rank 3 with vertex (the kernel of the associated 6×6 matrix) the set defined by $a_{11} = a_{22} = a_{12} = 0$. This is no other than the linear space of conics containing the line $x = 0$.

The following proposition then comes as no surprise. The strategy of the proof is to show that all the concepts mentioned are well behaved under changes of coordinates in $\mathbb{P}^2$ and the induced changes of coordinates on our space $\mathbb{P}^5$.

3.6.2 Proposition. *Each point on Δ is singular on Σ with multiplicity 2. The (projective embedded) tangent cone to Σ at L^2 is the rank 3 quadric consisting of conics tangent to or containing L together with the double lines. The vertex of this quadric is the space Σ_L of conics containing L.*

Remark. Note that by 3.5.3 this implies that the vertex of the tangent cone to Σ at L^2 is the tangent space to Δ at L^2.

Proof. By the above discussion, the proposition holds for the conic defined by $x^2 = 0$. We now prove that everything is well behaved under a change of coordinates $T \in \mathbb{P}GL_3$ taking the line $x = 0$ to the line L.

We will refer to the induced change of coordinates $\phi_T : \mathbb{P}^5 \to \mathbb{P}^5$ on the space of conics by Φ for convenience. Any such change of coordinates leaves the rank of any quadratic form unchanged, so under Φ the hypersurface H goes to another quadric of rank 3. The vertex of the quadric also transforms nicely since the kernel the matrix associated to the quadratic form goes to the kernel of the new matrix. Therefore, the vertex of $\Phi(H)$ is Φ("conics containing $x = 0$"), which is just Σ_L. The same happens with conics tangent to $x = 0$ which get mapped to conics tangent to L and with double lines which get mapped to double lines (in particular $\Phi(p) = L^2$). We therefore obtain that $\Phi(H)$ is the space of conics tangent to or containing L together with the double lines.

We now prove that $\Phi(H)$ is in fact the tangent cone to Σ at L^2 and with this we finish the proof of the proposition. However, this follows from the fact that $\Phi(\Sigma) = \Sigma$ and the general fact that a linear change of coordinates in $\mathbb{P}^n$ takes the projective tangent cone of a hypersurface to the projective tangent cone of the image of the point at the image hypersurface. □

Remark. Using the description given in the proposition we give the probable equation of the tangent cone at L^2 in the next section (probable because there is a detail missing). The approach, besides incomplete, apparently does not lend itself to give a proof of the previous proposition.

3.7 Duality

The dual of a smooth hypersurface X defined by a form F in $\mathbb{P}^n$ is defined to be the set in the dual space $\mathbb{P}^{n*}$ of hyperplanes which are tangent to X at some point. As usual, we identify the dual of $\mathbb{P}^n$ with $\mathbb{P}^n$ itself and we then obtain that the dual of X is the image of the map $X \to \mathbb{P}^n$ defined by

$$p \mapsto [\partial F/\partial x_0\,(p) : \ldots : \partial F/\partial x_n\,(p)].$$

Since X is smooth we see that this map is a morphism and so the dual of X is closed. When X is not a hyperplane one can prove that the dual of X is in fact a hypersurface by showing that the above map is finite [3] and using the theorem on the dimension of fibers. In the case of conics one can give the equation quite explicitly:

3.7.1 Proposition. *The dual of a smooth conic is a smooth conic. Moreover, the dual of a conic corresponding to a matrix A is the conic corresponding to the matrix A^{-1}.*

Proof. We can write the equation of the conic by $v^t A v = 0$ where $v = [x : y : z]$. Now, if one writes down the map giving the dual variety one sees that is it given by

$$\phi : [x : y : z] \mapsto \left[A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]$$

[3]See [Sha94], Chap. *II*, exercise 21 to section 1, and Chap. *I* section 5.3 theorem 8.

(the actual matrix in the map is $2A$, but we may ignore the 2 because of the projective coordinates). Now, if we make a change of coordinates in the image space sending $[x:y:z] \mapsto A^{-1}[x:y:z]^t$ then ϕ becomes the identity. We therefore have an equation for the image modulo a change of coordinates, to say, the original one $v^tAv = 0$. Reversing the change of coordinates we see that the equation of the dual is given by $0 = (A^{-1}v)^tA(A^{-1}v) = v^tA^{-1}v$. $\square$

The previous proposition allows us to give a probable equation for the quadric in $\mathbb{P}^5$ of conics tangent to or containing a given line L (together with the double lines). The following argument is not complete, but will most likely work:

Fix a line $L = [l_0 : l_1 : l_2]$. We wish to describe the locus of conics to which L is tangent algebraically. Now, assuming that the conic corresponding to a fixed matrix A is smooth, this will be the case if and only if L is a point in the dual conic. By the proposition above this conic is given by A^{-1}, and so L is tangent to A if and only if $L^tA^{-1}L = 0$.

Writing out the general equation for A like

$$\begin{bmatrix} 2a_{00} & a_{01} & a_{02} \\ a_{01} & 2a_{11} & a_{12} \\ a_{02} & a_{12} & 2a_{22} \end{bmatrix},$$

we may rewrite the expression $L^tA^{-1}L$ as a function in the a_{ij}. Moreover, we have $A^{-1} = \mathrm{adj}(A)/\det(A)$ where $\mathrm{adj}(A)$ is the adjugate matrix of A, and so we can clear the denominator in the equation $L^tA^{-1}L = 0$ and get the polynomial equation $L^t\mathrm{adj}(A)L = 0$ in the a_{ij}. This equation now makes sense regardless of whether A corresponds to a smooth conic or not, and we still have that if A corresponds to a smooth conic then the L that satisfy this equation are the tangents to A.

We now find what the locus of matrices satisfying the equation $L^t\mathrm{adj}(A)L = 0$ for a fixed line L is. We identify the conic with its corresponding matrix in what follows. As explained before, if A is smooth then the equation holds if and only if L is tangent to A. If A has rank 1 (corresponding to a double line in Δ), then $\mathrm{adj}(A) = 0$ since the components of $\mathrm{adj}(A)$ are the minors of A, so this equation is trivially satisfied.

What remains to be proved (and the missing part of the argument) is that if A has rank 1 (corresponding to a union of two lines) then the equation is satisfied only for matrices corresponding to conics containing L.

The definition of the dual of a hypersurface can be extended to singular hypersurfaces by defining it to be the closure of the dual of the open smooth locus of the hypersurface. We can also extend it to varieties in general by the following: We define a hyperplane to be tangent to a smooth variety Y if it contains an (embedded) tangent plane to Y, and we define the dual of Y to be the locus of its tangent hyperplanes. Finally, for singular Y we define the dual of Y to be the closure of the dual of its smooth locus.

3.7.2 Proposition. *The dual of Σ is (trivially isomorphic to) Δ, and the dual of Δ is (trivially isomorphic to) Σ.*

Proof. For Σ we consider its smooth locus $\Sigma - \Delta$. Remember that in 3.5.1 we proved that the tangent space to Σ at $L \cdot M$ is the space of conics containing $p = L \cap M$. By the first equation of section 3.3 we see that if $p = [p_0 : p_1 : p_2]$ then this hyperplane corresponds in the dual space $\mathbb{P}^{5*}$ to the point $[p_0^2 : p_1^2 : p_2^2 : p_0p_1 : p_0p_2 : p_1p_2]$. Since any point $p \in \mathbb{P}^2$ is the intersection point of two distinct lines, we see that the dual variety of Σ is precisely the image of the Veronese map $\nu_2 : \mathbb{P}^2 \mapsto \mathbb{P}^5$, which as explained in proposition 3.1.1 is trivially isomorphic to Δ.

Regarding Δ, by proposition 3.5.3 we know that the tangent spaces to Δ are the spaces Σ_L of conics containing a given line L. We now prove that a hyperplane contains some Σ_L if and only if its coordinates in $\mathbb{P}^{5*}$ correspond (modulo multiplication by 2 in some entries) to a singular conic.

Let then $b = [b_{00} : \ldots : b_{12}] \in \mathbb{P}^{5*}$ be a hyperplane containing the space Σ_L where $L = [l_0 : l_1 : l_2]$. In particular b contains the conics corresponding to xL, yL and zL where by xL we mean the conic corresponding to the union of the lines $x = 0$ and L. In coordinates $xL = [l_0 : 0 : 0 : l_1 : l_2]$, and there are similar expressions for yL and zL. Plugging these points into the equation of b given by $b_{00}a_{00} + \ldots + b_{12}a_{12} = 0$ we get the equations

$$\begin{aligned} l_0b_{00} + l_1b_{01} + l_2b_{02} &= 0 \\ l_0b_{01} + l_1b_{11} + l_2b_{12} &= 0 \\ l_0b_{02} + l_1b_{12} + l_2b_{22} &= 0 \end{aligned}$$

which imply the vanishing of the determinant of the coefficients b_{ij} since the l_0, l_1, l_2 are not all zero. Sending $\mathbb{P}^{5*} \to \mathbb{P}^5$ by the map

$$[b_{00} : \ldots : b_{12}] \mapsto [b_{00} : b_{11} : b_{22} : 2b_{01} : 2b_{02} : 2b_{12}]$$

we see that b corresponds to a singular conic (i.e., a conic in Σ).

In the reverse direction, the inverse map $\mathbb{P}^5 \to \mathbb{P}^{5*}$ takes a conic in $\Sigma - \Delta$ to an element $b = [b_{ij}]$ such that the system of equations above has a non-zero solution $[l_0 : l_1 : l_2]$. Let L be the corresponding line. Then the arguments above show that the hyperplane corresponding to b contains the conics $xL = 0$, $yL = 0$ and $xL = 0$, where now we are denoting by L the equation defining L. Since the hyperplane is linear it therefore contains $L(ax + by + cz)$ for any $[a : b : c] \in \mathbb{P}^2$ and this is precisely Σ_L. □

3.8 Chordal Varieties

The Chordal Variety to a variety X is the locus of all lines joining any two distinct points of X. Our last result regarding the space of conics and the surprising relations between Σ and Δ is the following:

3.8.1 Proposition. *The chordal variety of Δ is Σ.*

Proof. Remember that the line through any two points in $p, q \in \mathbb{P}^n$ is the space spanned p and q, that is, it is the set of points in $\mathbb{P}^n$ of the form $\{[sp + tq] \mid [s : t] \in \mathbb{P}^1\}$.

Identify now the space $\mathbb{P}^5$ with the projective space of symmetric matrices. The line through two matrices A, B is then the set of matrices which are linear combinations of A and B. Now, if A and B have rank 1, then any linear combination of A and B has rank at most 2. Thus, the line joining any two points of Δ is certainly contained in Σ.

In the other direction we first start with a simple case and then obtain the general result by a change or coordinates.

Consider the conic $xy = 0$ in Σ and note that $(x-y)^2-(x+y)^2 = -4xy$. This implies that we can write the conic $xy = 0$ as a linear combination of conics that are double lines, that is $(x-y)^2 = 0$ and $(x+y)^2 = 0$ as follows: If we denote by $A_{xy}, A_{(x-y)^2}$, and $A_{(x+y)^2}$ the matrices corresponding to these conics then we have $-4A_{xy} = A_{(x-y)^2} - A_{(x+y)^2}$, so projectively $[A_{xy}] = [A_{(x-y)^2} - A_{(x+y)^2}]$. This proves that the point on Σ corresponding to the conic $xy = 0$ is on the line joining the points corresponding to $(x - y)^2 = 0$ and $(x + y)^2 = 0$. Therefore $xy = 0$ is on the chordal variety of Δ.

Let now $L \cdot M \in \Sigma$ be a general point which is not in Δ, let $T \in \mathbb{P}GL_3$ be a change of coordinates taking $x = 0$ to L and $y = 0$ to M, and let $\Phi = \Phi_T$ be the induced change of coordinates of $\mathbb{P}^5$. Then $\Phi(xy = 0) = L \cdot M$, $\Phi((x - y)^2 = 0), \Phi((x + y)^2 = 0) \in \Delta$ and since Φ is linear, the line joining $(x - y)^2 = 0$ and $(x + y)^2 = 0$ is taken to the line joining $\Phi((x - y)^2 = 0)$ and $\Phi((x + y)^2 = 0)$. This proves the proposition. □

Remark. Remember that we had already proved that Σ is also the tangent variety to Δ in 3.5.4. Thus, we now have that Σ is surprisingly both the locus of all the tanent tangent lines to Δ and the locus of all the lines joining two distinct points of Δ.

For more information regarding chordal varieties see [Har92].

3.9 Anything else?

There are some topics regarding the space of conics that were left out for lack of time or background. The following is a short list of topics that could be pursued further.

The singularity of Σ: We proved that Σ is singular along Δ with multiplicity 2 and found the tangent cone at each of its points. It is possible that much more could be said about the nature of these singularities, but this probably requires a deeper knowledge of singularities of higher dimensional varieties.

Families of Conics This topic was barely discussed and there are lots of interesting issues. Examples include

- Conditions imposed by degenerate configurations of points: We proved that if no four of five points are collinear then there is a unique conic passing through those points, but we did not prove that the consition was necessary. Are there any other configurations that imply uniqueness? What about spaces of conics defined by configurations of fewer points?

- Pencils of conics: These are lines in $\mathbb{P}^5$. One can prove that there are exactly three singular conics in a pencil if and only if any two conics in this family intersect transversely (see [Har92] proposition 22.34 and its preceding discussion). This obviously has to do with the fact that Σ is of degree 3 which implies that a general line will only intersect Σ at three points.

- Conics tangent to five lines: The following is taken from the Wikipedia article on Enumerative Geometry which references William Fulton's book on Intersection Theory. It would be nice to fill in the details: *William Fulton gives the following example: count the conic sections tangent to five given lines in the projective plane. The conics constitute a projective space of dimension 5, taking their six coefficients as homogeneous coordinates. Tangency to a given line L is one condition, so determined a quadric in $\mathbb{P}^5$. However the linear system of divisors consisting of all such quadrics is not without a base locus. In fact each such quadric contains the Veronese surface, which parametrizes the conics $(aX + bY + cZ)^2 = 0$ called 'double lines'. The general Bzout theorem says* 5 *quadrics will intersect in* $32 = 2^5$ *points. But the relevant quadrics here are not in general position. From* 32, 31 *must be subtracted and attributed to the Veronese, to leave the correct answer (from the point of view of geometry), namely* 1.

4 The Space of Plane Cubics

As explained in the Introduction, my original intention was to include in this section an analysis of the space of plane cubics similar to the one above for plane conics. However, I ran both out of time and space. I include some comments about how interesting this space might be.

In this case $N = 9$ and the parameter space will contain far more varieties of interest. On the one hand, singular cubics no longer need to be reducible, and the singularity may be either a node or a cusp. On the other hand, reducible cubics can be all sorts of things. The full list of possibilities is given bellow.

Singular Cubics:

· Reducible.
· Irreducible with a nodal singularity.
· Irreducible with a cuspidal singularity

Reducible Cubics:

· A conic and a tangent line.
· A conic and a line not tangent to the conic.
· Three distinct lines meeting at three distinct points.
· Three lines meeting at a point.
· A double line and another line.
· A triple line.

By the contents of section 2 both the loci of reducible and singular conics are closed in $\mathbb{P}^9$. Moreover, by the same sort of arguments as the ones given for conics one sees easily that the locus of most of the entries in these two lists are closed sets in $\mathbb{P}^9$. For a pictorial depiction of the lattice of containments of each of these varieties see [Har92] example 10.16.

It should be noted that besides not having the theory of quadratic forms at our disposal to help us out in this case, the complexity of the equations involved gets completely out of hand. A striking example is the equation defining the locus of singular cubics (remember the existence of this equation was proved in 2.3.3). This equation is found by taking the discriminant of a general equation of a cubic in the most general Weierstrass form and then unwinding the changes of coordinates that made the equation so nice.

I would have included the polynomial here if it did not take up 10 and a half (yes, `ten` and a half) pages to display in inline mode with this font size and these margins. To see this polynomial the reader may either email me for a copy including it, or see [RVTA05] to reproduce it. There is also a more pleasing description of this polynomial as the determinant of a 6×6 matrix with entries that are homogeneous in the coefficients. See [Poo01] for this description.

Another important thing to notice is that in this case the curve no longer defines the defining polynomial uniquely, as seen for example with x^2y and xy^2 which both define the same curve.

A good place to start the study of the space of cubics would be to read the 1932 article [Yer32].

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