The Tangent Space to a Scheme

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Spring 2011

Definition Let X be an arbitrary scheme and $p \in X$ a point. We define the tangent space to X at p to be the dual space of the $\kappa(p)$-vector space m_p/m_p^2 where m_p is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

Tangent space to a k-rational point (Hartshorne Ex II.2.8) Let X be a scheme over a field k. Show that giving a morphism of k-schemes

$$\phi : \text{Spec}\left(k[\epsilon]/\langle\epsilon^2\rangle\right) \to X$$

is exactly the same as giving a point $p \in X$ with $\kappa(p) = k$ and an element of T_p.

An intuition why this whole thing works is the following (from Math Stack Exchange Qiaochu Yuan):

> Let $k[V] = k[x_1, ...x_n]/(f_1, ...f_r)$ be the ring of functions on the variety $V = \{f_1 = ... = f_r\} = 0$. Then a homomorphism $\phi : k[x_1, ...x_n]/(f_1, ...f_r) \to k[\epsilon]/(\epsilon^2)$ is precisely determined by the images $\phi(x_i) = p_i + \epsilon qi \in k[\epsilon]/(\epsilon^2)$ subject to the condition that $f_j(p + \epsilon q) = 0$.
>
> The key point here is that
>
> $$f_j(p + \epsilon q) = f_j(p) + \epsilon \sum_i q_i \partial f_j / \partial x_i(p)$$
>
> This is just truncated Taylor expansion, and the corresponding statement is true for $k[\epsilon]/(\epsilon^n)$ for every finite n. Hence this condition holds if and only if the pi define a point of V and the q_i define a vector orthogonal to the gradients of each of the f_j; this is precisely the condition that they define a tangent vector over $\mathbb{R}$ so we adopt it as our definition of tangent vector in general. It's not hard to see that we can add tangent vectors.
>
> The equations above allow us to give the following definition of the tangent bundle: define the polynomials $g_j = \sum_i y_i \partial f_j / \partial x_i(x) \in k[x_1, ...x_n, y_1, ...y_n]$, and then the tangent bundle ought to be Spec $k[x_1, ...x_n, y_1, ...y_n]/(f_1, ...f_r, g_1, ...g_r)$. There is probably a coordinate-independent way to state this definition.

Solution: Let p be the image of the point $\star \in T = \operatorname{Spec}\left(k[\epsilon]/\langle\epsilon^2\rangle\right)$ under the morphism. Then we get and induced map on the stalks

$$\phi_p : \mathcal{O}_{X,p} \to \mathcal{O}_{T,\star} = \left(k[\epsilon]/\langle\epsilon^2\rangle\right)_{\langle\epsilon\rangle} = k[\epsilon]/\langle\epsilon^2\rangle$$

where the description of the stalk of T comes from the previous set of exercises. Since this map is local then it induces an inclusion of fields

$$\kappa(p) = \mathcal{O}_{X,p}/m_p \hookrightarrow \mathcal{O}_{T,\star}/m_\star = k. \qquad (*)$$

Moreover, since the map ϕ is a morphism of schemes over k then the commutativity of the diagram

$$\begin{array}{ccc} T & \stackrel{\phi}{\to} & X \\ & & \\ \searrow & \# & \swarrow \\ & & \\ & \operatorname{Spec} k & \end{array}$$

gives the commutativity of the maps on the stalks

$$\begin{array}{ccc} k[\epsilon]/\langle\epsilon^2\rangle & \stackrel{\phi_p}{\longleftarrow} & \mathcal{O}_{X,p} \\ & & \\ \nwarrow & \# & \nearrow \\ & & \\ & k & \end{array} \qquad (**)$$

as rings. Putting this together with the map $(*)$ above gives the commutativity of the following map of rings

$$\begin{array}{ccc} k & \hookleftarrow & \kappa(p) \\ & & \\ \uparrow & \# & \uparrow \\ & & \\ k[\epsilon]/\langle\epsilon^2\rangle & \stackrel{\phi_p}{\longleftarrow} & \mathcal{O}_{X,p} \\ & & \\ \nwarrow & \# & \nearrow \\ & & \\ & k & \end{array} \qquad (***)$$

where the vertical maps on the left of the diagram are the inclusion and the quotient map by the maximal ideal $\langle\epsilon\rangle$ given by

$$\begin{array}{ccccc} k & \to & k[\epsilon]/\langle\epsilon^2\rangle & \to & k \\ a & \mapsto & a & & \\ & & a + b\epsilon & \mapsto & a \end{array}$$

Thus, we get the inclusion $k \hookrightarrow \kappa(p) \hookrightarrow k$ (following the right of the diagram) so that $\kappa(p) \simeq k$. This concludes the proof that p is a k-rational point.

Now for the fact that we get an element of the tangent space, note that since ϕ_p is local then it maps the maximal ideal of $\mathcal{O}_{X,p}$ to the maximal ideal of $k[\epsilon]/\langle\epsilon^2\rangle$ so restriction of ϕ_p gives the map

$$m_p \xrightarrow{\phi_p} m = \langle\epsilon\rangle.$$

This map is k-linear with multiplication by k given by inclusions in $(**)$ because the diagram is commutative and ϕ_p is a ring homomorphism. Since $\epsilon^2 = 0$ this induces a k-linear map

$$m_p/m_p^2 \to m_\star = \langle\epsilon\rangle$$

where the action of k on the right is given by multiplication since this is the k-algebra structure we gave $k[\epsilon]/\langle\epsilon^2\rangle$, and since $\langle\epsilon\rangle$ is naturally isomorphic to k as a k-vector space we finally get a k-linear map

$$m_p/m_p^2 \to k \qquad (\diamondsuit)$$

where the action of k on m_p/m_p^2 is induced by the structure map $X \to \text{Spec } k$.

We are very close to getting something in the tangent space, since

$$T_{X,p} = \left(m_p/m_p^2\right)^* = \text{Hom}_{\kappa(p)}(m_p/m_p^2, \kappa(p)),$$

and already know that $\kappa(p) \simeq k$. The $\kappa(p)$-module structure on m_p/m_p^2 used in the definition of the tangent space above is given by multiplication of elements of $\mathcal{O}_{X,p}/m_p$ (explicitly, if $r \in m_p/m_p^2$ and $a \in \kappa(p)$, then $a \cdot r = sr$ where $s \in \mathcal{O}_{X,p}$ is any lift of a by the quotient map $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/m_p$), and one could expect that the $k(p)$-module structure on m_p/m_p^2 agrees with the k-module structure that we used in the map $m_p/m_p^2 \to k$ from $(\diamondsuit)$ above, but this is in fact not the case. However, we can use the k-linear map in $(\diamondsuit)$ to obtain an element in $\text{Hom}_{\kappa(p)}(m_p/m_p^2, \kappa(p))$ in a natural way by using the k-scheme structure of X. The details are as follows:

The k-scheme structure morphism $X \to \text{Spec } k$ induces the maps

$$k \to \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/m_p = \kappa(p)$$

which form the right vertical column of $(***)$. Since we already know that $\kappa(p) \simeq k$, this gives a canonical isomorphism of k with $\kappa(p)$ coming from the k-scheme structure of X. Let h be the inverse of this composition, and label the other maps as follows:

$$k \xrightarrow{f} \mathcal{O}_{X,p} \xrightarrow{g} \kappa(p) = \mathcal{O}_{X,p}/m_p, \qquad h : \kappa(p) \to k \qquad (\spadesuit)$$

Using the isomorphism $h : \kappa(p) \to k$ we can construct and induced $\kappa(p)$-linear map from $(\diamondsuit)$

$$m_p/m_p^2 \to \kappa(p)$$

by giving m_p/m_p^2 a $\kappa(p)$-vector space structure using h. Specifically, if $a \in \kappa(p)$ and $r \in m_p/m_p^2$ then we define $a \cdot r := h(a) \cdot r$ where the action on the right is given by the k-vector space structure used in ($\diamondsuit$), and then the map is defined by ($\diamondsuit$). By definition, the action used in ($\diamondsuit$) is given by $b \cdot r = f(b)r$ where $f(b) \in \mathcal{O}_{X,p}$ and the product is multiplication in $\mathcal{O}_{X,p}$. Putting this together we see that the induced action of $\kappa(p)$ we just defined in m_p/m_p^2 is given by $a \cdot r := h(a) \cdot r = f(h(a))r$, but by ($\spadesuit$) we have $g(f(h(a))) = a$ since $g \circ f$ is the inverse of h and so $f(h(a))$ is a lift of a to $\mathcal{O}_{X,p}$. This implies that the induced action of $\kappa(p)$ we just defined on m_p/m_p^2 is precisely the canonical action of $\mathcal{O}_{x,p}/m_p$ on m_p/m_p^2!

Thus, the $\kappa(p)$-linear map is an element of $\mathrm{Hom}_{\kappa(p)}(m_p/m_p^2, \kappa(p))$ with the canonical $\kappa(p)$-vector space structure on m_p/m_p^2 and so an element of T_p. This concludes the proof that a morphism of k-schemes $\phi : \mathrm{Spec}\left(k[\epsilon]/\langle\epsilon^2\rangle\right) \to X$ gives a point $p \in X$ with $\kappa(p) = k$ and an element of T_p.

In the reverse direction, take $p \in X$ with $\kappa(p) = k$ and a point $v \in T_p$. Let the isomorphism of $\kappa(p)$ with k be given by the induced map from the k-scheme structure $k \to \mathcal{O}_{x,p} \to \kappa(p) = \mathcal{O}_{X,p}/m_p$.

We can view v as a $\kappa(p)$-linear map $v : m_p/m_p^2 \to \kappa(p)$. Define the map

$$\begin{array}{rcl} \psi : \mathcal{O}_{X,p} & \to & k[\epsilon]/\langle\epsilon^2\rangle \\ r & \mapsto & h(r(p)) + v(r - f(h(r(p))))\epsilon \end{array}$$

where $r(p) = g(r)$ is the "value" of r at p and we using the notation from ($\spadesuit$) (in the case when X is an affine variety and $\mathcal{O}_{X,p}$ is a space of functions this $r - f(h(r(p)))$ is really $r - r(p) \in m_p$, but in this very general setting we have to use all these maps to make sense of the value of r at p as an element of $\mathcal{O}_{X,p}$). Once we know that this is well defined and a ring homomorphism then we can define the morphism $\phi : T \to X$ by sending $\star$ to p and the morphism on the sheaves by sending $\mathcal{O}_X(U) \to \mathcal{O}_{x,p} \to k[\epsilon]/\langle\epsilon^2\rangle$ if $p \in U$ (we don't need to define anything if $p \notin U$ since then the inverse image of U is empty!) and this will immediately be a morphism. Finally, we need to check that ϕ is a morphism of k-schemes.

ψ IS WELL DEFINED AND A RING HOM: We first need to check that $r - f(h(r(p))) \in m_p$. Now, by ($\spadesuit$) we have

$$\begin{array}{rcl} g(r - f(h(r(p)))) & = & g(r) - g(f(h(r(p)))) \\ & = & g(r) - r(p) \\ & = & 0 \end{array}$$

since $r(p) = g(r)$, g is a ring homomorphism and $f \circ g$ is the inverse of h. This implies that $r - f(h(r(p))) \in m_p$.

To see why ψ is a ring homomorphism note that

$$v(rs - f(h(rs(p))) = v(r - f(h(r(p))))s(p) + v(s - f(h(s(p))))r(p)$$

since by k linearity this is equivalent to

$$v(rs - f(h(rs(p))) = v([r - f(h(r(p)))]f(h(s(p)))) + v([s - f(h(s(p)))]f(h(r(p))))$$

and this is true since the difference of the inputs is $(r - f(h(r(p))))(s - f(h(s(p)))) \in m_p^2$. This implies that $\psi(rs) = \psi(r)\psi(s)$. The rest of the proof that ψ is a ring homomorphims is then simple. One also checks that ψ is a k-algbera map where the k-algebra structure on $\mathcal{O}_{X,p}$ comes from the $k$scheme structure.