The 27 lines on a smooth cubic surface and the 28 bitangents on a smooth quartic curve.

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This comes from an exercise in Beltrametti, Carletti, Gallarati, Monti 's book "Lectures on Curves, Surfaces and Projective Varieties". It is exercise 5.8.13.

First let S be a smooth cubic surface in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ and assume that $p = [1 : 0 : 0 : 0] \in S$. This last assumption implies that we can write the equation of S in the form

$$f = x_0^2\phi_1(x_1, x_2, x_3) + 2x_0\phi_2(x_1, x_2, x_3) + \phi_3(x_1, x_2, x_3) = 0.$$

Let C be the curve in $\mathbb{P}^2_{x_1,x_2,x_3}$ with equation

$$\Delta = \phi_1\phi_3 - \phi_2^2,$$

and note that Δ is the discriminant of f when viewed as an element of $k[x_1, x_2, x_3][x_0]$, and so the points $q \in C$ are precisely to q for which $f(x_0, q)$ is a square in $k[x_0]$. Note furthermore that ϕ_i is homogeneous of degree i and so C is a quartic (degree 4) in $\mathbb{P}^2$.

Let now V be the cone in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ defined by Δ, or in other words, consider C inside $\{x_0 = 0\}$ and let V be the set of all the lines joining C to p.

Since S is a cubic surface, any line in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ intersects S at 3 points counting multiplicities, and the important fact about the lines generating V is that the line joining a point $q \in C$ to p will intersect X at p and at another point with multiplicity 2. To see this explicitly, let $q \in C$ and let L be the line joining q and p. Parametrically L is given by

$$sp + tq = [s : tq_1 : tq_2 : tq_3]$$

and its intersection with S is give by the roots of the equation $f(L) = 0$ which gives

$$s^2t\phi_1(q) + 2st^2\phi_2(q) + t^3\phi_3(q) = 0$$

(because ϕ_i is homogeneous of degree i). One root is $t = 0$ which corresponds to p, and the other is a double root because the discriminant of

$$\left(\frac{s}{t}\right)^2 \phi_1(q) + 2\left(\frac{s}{t}\right)\phi_2(q) + \phi_3(q) = 0$$

vanishes. Thus, all the lines that generate V have a point of double intersection with S.

For the rest of the argument we need to make an extra assumption on C: we need to assume that C is smooth so that it has 28 bitangents. It is from these bitangents that we will obtain the 27 lines on S, and so from now on assume C is smooth, which will happen for general S.

Say L is a bitangent to C through the two points q_1, q_2. Then the cone over L is a plane π in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ which intersects S at two points with multiplicity 2 (the special points lying above q_1 and q_2). Therefore, the intersection of π with S is a cubic curve with two double points. But a cubic curve in $\pi \cong \mathbb{P}^2$ can have two double points only if it is reducible, with one component being the line going through the two double points. Thus, we see that $\pi \cap S$ contains a line, and that moreover this line gets projected down onto L.

Now this argument can't work in general because S would contain 28 lines and not 27. The point is that ϕ_i is homogeneous of degree i and so $\phi_1 = 0$ is a bitangent to C because its intersection with C is given by the equation $\phi_2^2 = 0$ which will intersect $\phi_1 = 0$ at two double points, so it will intersect C with multiplicity 2 at two points. The plane π above $\phi_1 = 0$ does not satisfy what we were saying above for planes above bitangents to C because if q is one of points of tangency of $\phi_1 = 0$ with C, then the intersection of the line through q and p is given by the roots of

$$2st^2\phi_2(q) + t^3\phi_3(q) = 0$$

(since $\phi_1(q) = 0$) which implies that $t = 0$ corresponding to p is a double root.

Apart from this special bitangent, the other 27 do each give a line on the cubic surface! One needs to argue that all the lines are distinct, and the point is that each line projects down to its the corresponding bitangent, i.e., if L was cut by a plane π above the bitangent to C through the two points q_1, q_2, then L contains the two double points, and so has points above both q_1 and q_2 which implies that it gets projected down to the whole bitangent. Since a bitangent only intersects the curve at its two points of tangency because the degree of C is 4, then all 28 bitangents are distinct and so there are 27 lines!

Conversely, if L is a line on S and we assume that p is not contained in any line, then the plane spanned by p and L intersects S in a reducible cubic with two double points on L. Therefore, the lines joining p to these double points will only intersect S at two points, and so the projection of these lines will be two points in C. The line joining these points, which is the projection of the plane pL will be a bitangent to C because it only intersects C at these two points (one needs to argue that the intersections are both double), and so this implies that the plane pL was one of the 28 we considered before.

This concludes the explanation of the relation between the 27 lines and the 28 bitangents of plane quartics! It is far from being a proof because there are details to be supplied, things to be checked, and it only works for general S. Nonetheless, it is nice to see that there is in fact a clear relation between the two counts, and specially why out of the 28 bitangents only 27 correspond to lines on the cubic.