Topology of Plane Algebraic Curves

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1 The Real Projective Plane $\mathbb{RP}^2$

In this section we will introduce all the terminology and some pictures that help to think about the real projective plane. Our main goal is to talk about curves in the complex projective plane, but it is of course easier to get some intuition on a space which is two dimensional even if it cannot be embedded in three dimensional real space in a satisfactory way.

Remember that the real projective plane $\mathbb{RP}^2$ is defined to be the set $\mathbb{R}^3 - \{(0,0,0)\}$ modulo the equivalence relation $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for $\lambda \in \mathbb{R}^\times$. The points in this space will be denoted by $[x : y : z]$, where of course x, y, z are not all zero. There are various ways to think about this space and I will review some of them below.

$\mathbb{RP}^2$ is the set of lines of $\mathbb{R}^3$ going through the origin. This is almost immediate by the definition. The equivalence relation identifies any two points that lie on the same line in $\mathbb{R}^3$. This is usually the least useful way to think about the projective plane.

$\mathbb{RP}^2$ is a sphere S^2 with diametrically opposite points identified. This follows by taking the representatives of each line lying on the sphere. Given any line there will be two points of it on the sphere, and they will be diametrically opposite.

$\mathbb{RP}^2$ is a disc D^2 with opposite points on its boundary circle identified. This follows by just taking the lower-hemisphere representative for each equivalence class above excluding the equivalence classes on the equator. This is as close as we can get to a picture of the real projective plane looking like a plane. It allows us to see that $\mathbb{RP}^2$ is in fact two dimensional, and we have just one point in the picture for most of the points in $\mathbb{RP}^2$, the only ones remaining to be identified being the ones on the boundary.

$\mathbb{RP}^2$ can be stereographically projected onto the plane (excluding the points on the boundary) Place the lower hemisphere on top of the plane, and stereographically project from the center of the sphere. The points on the boundary will have no image, but we do get some intuition on what they stand for: they represent the directions in the plane. See the Figure 1.

$\mathbb{RP}^2$ is three real planes, all glued together in a very complicated way so that each one contains most of the other two. This follows from the fact that in the set where $x \neq 0$, the map

$$\begin{aligned} \mathbb{RP}^2 &\to \mathbb{R}^2 \\ [x : y : z] &\mapsto (y/x, z/x) \end{aligned}$$

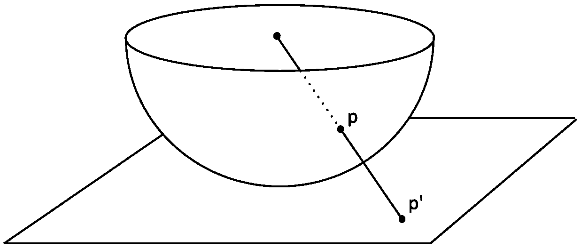

Figure 1: Stereographic projection of the real projective plane

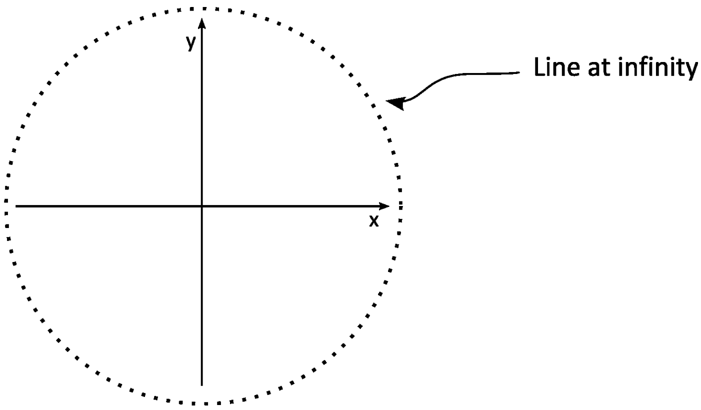

Figure 2: The plane and the line at infinity

is a homeomorphism (the topology of $\mathbb{RP}^2$ being the quotient topology), and the same happens on the sets where $y \neq 0$ and $z \neq 0$. If we call these spaces U_x, U_y, U_z respectively, we see that what each is missing from any other is a line (as will be explained in more detail in the next section). For example, U_z is missing the line $z = 0$. The line that each one of these is missing is called the line at infinity. One usually works with U_z, knowing that there is a line infinitely far away and not in the picture. See Figure 2.

As a matter of fact, the disc model we previously mentioned is precisely a bounded picture of the situation with the line at infinity being the boundary circle of the disc. Each point of the line at infinty fixes a direction.

We give a final way to view the projetive space. This is of little practical use for our purposes, but it is nonetheless interesting to note how weird the topology of $\mathbb{RP}^2$ is.

$\mathbb{RP}^2$ is the result of sewing the cirle boundary of a disc D^2 to the boudary of a Möbius band (a circle also). Take the square representation of the Möbius band where the two vertical segments are to be identified. We plan to sew the boundary of the disc to the top and bottom segments. To acomplish this, cut the square in half with a horizontal line, then flip the bottom part and identify two of the vertical segments that

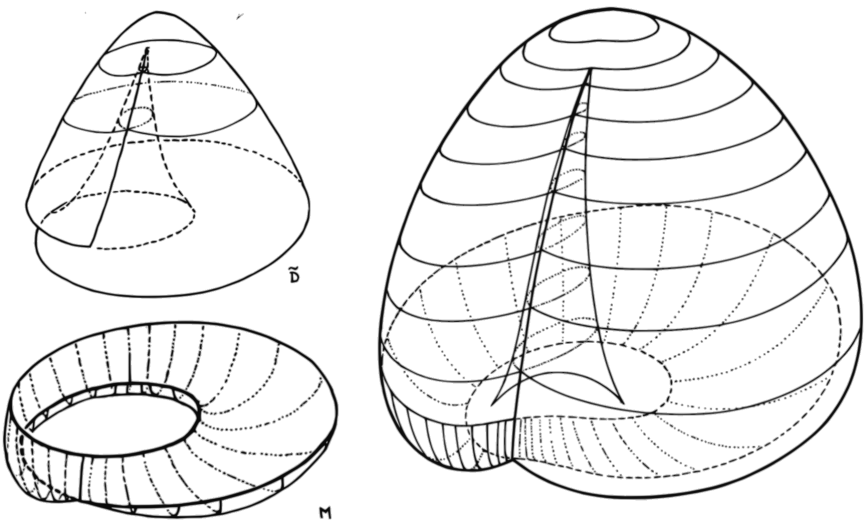

Figure 3: The Möbius band in the Real projective plane.

now match. To identify the remaining vertical edges bend the strip into an annulus making sure that the sides that have to be identified due to the cut we performed on the square form the outer cirle. The disc can now be sewn to the figure as the segments where we wanted to sew it to now form the inner disc of the annulus (with some pictures one may check that the orientations are correct). We then get a disc ojn which the diametrically opposite points of the boundary remain to be identified, that is, $\mathbb{RP}^2$.

Amazingly, one can draw a picture of $\mathbb{RP}^2$ depicting this fact! See the Figure 3 (taken form [1]).

1.1 Real Manifold Structure

The real projective plane has the structure of a real smooth manifold where the standard coordinate charts are the affine subspaces U_x, U_y, U_z considered earlier. The transition funtions are easily checked to be smooth, and some extra work shows that $\mathbb{RP}^2$ is moreover a non-orientable smooth manifold. The following theorem gives one of the reasons why one wants to study curves in the projective plane. Seeing a curve in the projective plane "compactifies" the curve.

Theorem 1. $\mathbb{RP}^2$ *is compact.*

Proof. It is the topological quotient of a compact space (the sphere). ∎

Any line of $\mathbb{RP}^2$ is given by an equation of the form $ax+by+cz = 0$ where not all a, b and c are zero. They are the result of viewing lines in $\mathbb{R}^2$ inside $\mathbb{RP}^2$ when $\mathbb{R}^2$ is identified with U_z or any other standard chart. Specifically, taking a line $dx + ey = f$ in $\mathbb{R}^2$ gives the equation $d(x/z) + e(y/z) = f$

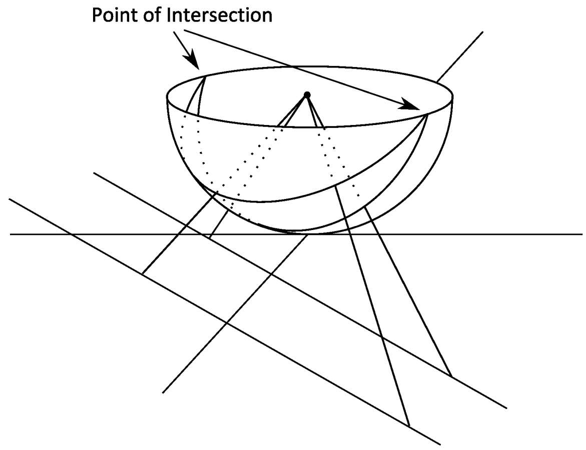

Figure 4: Intersection of Parallel lines at Infinity.

which we can transform to $dx + ey = fz$. This adds one extra point to the line, the one satisfying $z = 0$. Therefore, we see that, topologically, lines in $\mathbb{RP}^2$ are circles. They are the one point compactification of $\mathbb{R}$. A picture of this can be seen by using the stereographic projection: the points of the line lying on the equator of the sphere get identified giving a circle.

The following theorem is just a very particular case of Bezout's theorem, which states that any two projective curves of degrees n and m intersect exactly at nm points counting multiplicities. This is only true over the complex numbers, and one needs to define multiplicities of intersections of curves very carefully. The proof for intersections of lines is pretty simple.

Theorem 2. *Any two distinct lines in $\mathbb{RP}^2$ intersect at one point.*

Proof. Taking two distinct lines $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ where $[a_1, b_1, c_1] \neq [a_2, b_2, c_2]$ we see that the homogeneous system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \end{aligned}$$

has rank 2, and so has a one dimensional solution space in $\mathbb{R}^3$, the line of intersection of the planes. This is one point in $\mathbb{RP}^2$. ∎

Note that Theorem 2 implies that parallel lines on the plane intersect at one point when considered in the projective plane (the point at infinity). Bezout's theorem is even true for lines in $\mathbb{R}^2$! A very informal picture-proof of the previous result for the delicate case in which the lines are parallel can be given by means of a picture. It involves looking at the curves under the inverse stereographic projection explained above. Hopefully, Figure 4 will speak for itself.

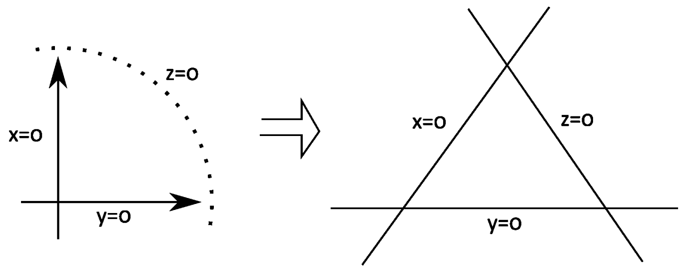

Figure 5: The three Axis of the Projective Plane

Theorem 2 also gives us one final way to view the projcetive plane. In $\mathbb{RP}^2$ there are three coordinate axis: the x, y and z axis. Any two of them appear in each one of the standard charts U_x, U_y and U_z, the missing one being the "line at infinity" of the corresponding chart. For example, the part of $\mathbb{RP}^2$ which is not present in U_z is the line defined by $z = 0$. We can schematically draw these three axis in the plane, bearing in mind Theorem 2 which implies that each coordinate axis intersects the other two. This gives the sketch of $\mathbb{RP}^2$ shown in Figure 5. Each line has the topological type of a circle, and if we consider a strip around each line with the line as the center circle we will get a Möbius band (to see this take a horizontal strip in the disc model of $\mathbb{RP}^2$ to get a band with the opposite edges identified in the way they are in the Möbius band).

2 Topology of Curves over the Complex Projective Plane

As was mentioned earlier, and by the nature of the results that can be obtained (eg. Bezout's Theorem), it is desirable to consider the base field to be $\mathbb{C}$ instead of $\mathbb{R}$. Just to give some extra examples justifying this change, note that in this setting the "one point curve" $x^2 + y^2 = 0$ over $\mathbb{R}^2$ ceases to be an aberrant "curve" and becomes the union of the lines $x = iy$ and $x = -iy$ since $x^2 + y^2 = (x + iy)(x - iy)$ over $\mathbb{C}$. Also, the line $x = 2$ will intersect the circle $x^2 + y^2 = 1$ in the two points it will be expected to.

The main topological fact about curves over the complex projective plane is that they are Riemann Surfaces as long as they are nonsingular (or smooth). This is what we will first discuss.

2.1 Riemann Surfaces

A Riemann surface is defined just like a differentiable manifold, there being three essential differences:

- The charts (ϕ, U) are maps $\phi : U \to \mathbb{C}$.
- The transition maps $\phi \circ \psi^{-1}$ have to be holomorphic on their domain ($\psi(U \cap V)$ if the carts are (ϕ, U) and (ψ, V)).
- The topological space is assumed to be connected.

Of course, the term "surface" used to refer to these objects comes from viewing them as real manifols. A basic fact following from the definition of Riemann surface is the following:

Theorem 3. *Every Riemann surface is orientable.*

Proof. To check orientability we have to look at the induced real 2-manifold structure of our Riemann surface. Let z and w be local coordinates on some overlapping domains. The requirement on the transition maps implies that there exists a holomorphic function f such that $f(z) = w$. If we let $z = x+iy$, $w = s+it$ and $f = u+iv$, then our local coordinates over $\mathbb{R}$ are given by (x, y) and (s, t) and the transition function is $f(x, y) = (u(x, y), v(x, y)) = (s, t)$. The Jacobian of this transformation is

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

whose determinant is (by the Cauchy Riemann equations) equal to $u_x^2 + v_x^2 = |f'(z)|^2 \geq 0$. Since f is biholomorphic by hypothesis (it is a change of basis and so the opposite change of basis, f^{-1}, is holomorphic also) we get that $f'(z) \neq 0$ by the chain rule. Thus, all the carts are compatible! ■

We will also need the following theorem regarding compact Riemann surfaces (the proof is not included).

Theorem 4. *Every compact Riemann surface is a g-holed torus.*

The number of holes is called the genus of the Riemann surface.

2.2 The Complex Projective Line $\mathbb{CP}^1$

This is the simplest curve over $\mathbb{C}$ that one can imagine. However, the reason why it is what we want to call "a curve" will only be clear until we get to discuss the projective plane $\mathbb{CP}^2$.

The construction of $\mathbb{CP}^1$ is no different from what was discussed before for $\mathbb{RP}^2$, but we go over it because we want to discuss some topological facts. $\mathbb{CP}^1$ is the quotient $\mathbb{C}^2 - \{0\}/\sim$ where $(x, y) \sim (kx, ky)$ for any $k \in \mathbb{C}^*$. Its points are denoted by $[x : y]$, and it can be covered by two particularly nice open sets U_x, U_y where $x \neq 0$ and $y \neq 0$ respectively. In a way similar to what was discussed for $\mathbb{RP}^2$, each U_x and U_y can be seen to be homeomorphic to $\mathbb{C}$ (e.g., $U_x \to \mathbb{C} : [x : y] \mapsto y/x$ is a homeomorphism), and so in this case we could say that $\mathbb{CP}^1$ is the union of two planes glued together in a very weird way. However, as we are about to see, the glueing is not that weird after all.

First, note that by identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ in the obvious way we get a norm on $\mathbb{C}^2$ given by

$$||(x, y)|| = ||(a + ib, c + id)|| = \sqrt{a^2 +^2 +c^2 + d^2}$$

For the construction of $\mathbb{CP}^1$ we can therefore restrict our attention to the points with norm 1 which give the 3-sphere $S^3(\mathbb{C})$ (the $\mathbb{C}$ is there only to point out that we are regarding it a a subset of $\mathbb{C}^2$). We then get that $\mathbb{CP}^1$ is the quotient $S^3(\mathbb{C})/\sim$ where $\sim$ identifies the points (x, y) and $(\lambda x, \lambda y)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Of course, this is no longer as simple as identifying "diametrically opposite points" (in some sense this is the quotient S^3/S^1). Nonetheless, this fact helps us understand why $\mathbb{CP}^1$ is compact.

Theorem 5. $\mathbb{CP}^1$ *is compact and connected.*

Proof. Let $(x, y) \in S^3(\mathbb{C}) \subseteq \mathbb{C}^2$. If $x = a + ib$ and $y = c + id$, we get

$$1 = |x|^2 + |y|^2 = a^2 + b^2 + c^2 + d^2,$$

and so, identifying $\mathbb{C}^2$ with $\mathbb{R}^4$, we see that $S^3(\mathbb{C})$ is the set where the continuous norm function

$$||(a, b, c, d)|| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

takes the value 1. As such, it follows that $S^3(\mathbb{C})$ is connected, closed and bounded (so compact). Since $\mathbb{CP}^1$ is $S^3(\mathbb{C})/\sim$, it follows that $\mathbb{CP}^1$ is also compact and connected (any quotient of a connected compact space will be so). ■

However, a closer look at the original definition of $\mathbb{CP}^1$ gives us much more than this. $\mathbb{CP}^1$ is in fact a very well known Riemann surface. The charts U_x and U_y cover $\mathbb{CP}^1$ and each is homeomorphic to $\mathbb{C}$. If we denote by $z = y/x$ and $z' = x/y$ the local coordinates of these charts, we see that the change of coordinates is given by $z' = 1/z$. This is holomorphic in (the image of) $U_x \cap U_y$. Moreover, if we identify U_x with $\mathbb{C}$, we see that to get $\mathbb{CP}^1$ we identify the complement of the origin of $\mathbb{C}$ with another copy of $\mathbb{C}$ by means of the formula $z = 1/z'$. That is, what do to obtain $\mathbb{CP}^1$ we just add one extra point to $\mathbb{C}$ (the origin of the other copy) at infinity of our plane ($|z| \to \infty$) and so $\mathbb{CP}^1$ is no other than the Riemann sphere S^2! The explicit formulas are given below.

Theorem 6. *As a differentiable manifold, $\mathbb{CP}^1$ is just S^2.*

Proof. Take $S^2 \subset \mathbb{R}^3$ with equation $s^2 + t^2 + w^2 = 1$ and define the map $S^2 \to \mathbb{CP}^1$ by

$$(s, t, u) \mapsto [s + it : 1 - u]$$

Then the inverse mapping can be seen to be given by

$$[x : y] \mapsto \left(\frac{2Re(x\bar{y})}{|x|^2 + |y|^2}, \frac{2Im(x\bar{y})}{|x|^2 + |y|^2}, \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2} \right).$$

These formulas are obtained by using a sort of stereographic projection of the sphere with center at the origin to the planes $u = 1$ and $u = -1$ thinking about them as copies of U_x and U_y and taking into account the appropriate identifications ($u = -1$ corresponds to $y = 0$ and $u = 1$ corresponds to $x = 0$). ■

As a final note regarding this result, note that with the identification of $\mathbb{CP}^1$ with S^2, the projection map in the proof of Theorem 5 gives a map $S^3 \to S^2$ where the primage of every point is a circle. This map is known as the *Hopf fibration* and is of great interest.

2.3 The Complex Projective Plane $\mathbb{CP}^2$

The complex projective plane is defined exactly like $\mathbb{RP}^2$ was defined above, replacing $\mathbb{R}$ everywhere by $\mathbb{C}$. It again can be seen as the quotient of a sphere, but this time in higher dimensions. Specifically, $\mathbb{CP}^2$ is $S^5(\mathbb{C})/\sim$ where $\sim$ identifies the points (x, y, z) on the sphere (5 real dimensions) with all its multiples by complex λ with $|\lambda| = 1$. Again, this implies that $\mathbb{CP}^2$ is connected and compact. We also see that $\mathbb{CP}^2$ is four dimensional as a real manifold, so we certainly have no accurate pictures now!

If we denote the coordinates in $\mathbb{CP}^2$ by $[x : y : z]$ as was done with $\mathbb{RP}^2$, then we see that $\mathbb{CP}^2$ is covered by the open sets U_x, U_y and U_z where $x \neq 0$, $y \neq 0$ and $z \neq 0$ respectively. Each of these is homeomorphic to $\mathbb{C}^2$ in a natural way. Note also that the part of $\mathbb{CP}^2$ missing in U_z (for example) is $\{[x : y : 0] : x, y \in \mathbb{C}\}$ which is a copy of $\mathbb{CP}^1$.

$\mathbb{CP}^2$ is the space where we want plane algebraic curves to live in: we started off with the real plane and then added points at infinity obtaining in this way the projective plane, and then we wanted to allow our coordinates to live in $\mathbb{C}$. This is what we get. Everything makes sense, at least intuitively, as long as we consider the dimensions over $\mathbb{C}$ instead of $\mathbb{R}$. $\mathbb{C}$ is to be regarded just as a field, and not as a two dimensional plane (even though in this paper we will try to look at the topology of the situation over $\mathbb{R}$). The plane over this field is $\mathbb{C}^2$, and our "curves" are defined by equations $f(x, y) = 0$. To make our space compact (this will also make our curves compact as we will see later on), and to get a coherent theory of intersections of curves, we add points at infinity of our plane getting $\mathbb{CP}^2$ and regard the original plane $\mathbb{C}^2$ as the open subset U_z.

Since our coordinates are homogeneous, curves in $\mathbb{CP}^2$ are given by equations of the form $f(x, y, z) = 0$ where $f(x, y, z)$ is a homogeneous polynomial in x, y, z. The passage between curves in $\mathbb{CP}^2$ and $\mathbb{C}^2$ is as follows: If $f(x, y, z) = 0$ is a projective plane curve, we get a curve $f(x/z, y/z, 1) = 0$ in $\mathbb{C}^2$. In the other direction, starting with a curve $g(x, y) = 0$ in $\mathbb{C}^2$, we get its corresponding projective curve in $\mathbb{CP}^2$ by making $g(x, y)$ homogeneous by adding appropriate powers of z to each term (this is called the projective closure of the affine curve). Of course, there is nothing special about z in this whole discussion. We call the parts of a projective curve lying in U_x, U_y or U_z its affine parts. One can give a more general definition of the affine parts of a curve (showing there is nothing special about the sets where $x = 0$, $y = 0$ or $z = 0$ besides the fact that they are "lines") but this will not be necessary here.

We will now prove that every smooth projective curve in $\mathbb{CP}^2$ is in fact a compact Riemann surface.

2.4 Complex Projective curves are Riemann surfaces

Smoothness of plane projective curves is defined locally. That is, we say that a projective curve is smooth if every affine part is smooth. An affine curve is smooth if it is smooth at all of its points, and smoothness at a point is defined in terms of the multiplicity of the intersections of lines with the curve passing through that particular point. In most algebraic curves books one finds a proof of how this definition is equivalent to the following: an affine irreducible curve $f(x, y) = 0$ is smooth at a point (a, b) if the partial derivatives of f do not simultaneously vanish at (a, b).

One should notice that the curve $y^3 = x^4$ is not smooth at the origin even though the real picture looks smooth! (The reason being that it is only a C^1 submanifold of $\mathbb{R}^2$. See [1] pgs. 213-∞). One can see that a projective irreducible curve $F(x, y, z) = 0$ is smooth if and only if there is no point

$[a:b:c] \in \mathbb{CP}^2$ for which all the partial derivatives of F vanish (by Euler's theorem on derivatives of homogeneous polynomials) so that checking all the affine parts is rarely necessary. We are now ready to state the main theorem of this section. We begin with a lemma.

Lemma 7. *Any smooth affine curve $f(x,y) = 0$ is a (non-compact) Riemann Surface.*

Proof. Let $P = (a,b)$ be any point on the curve. The smoothness assumption implies that one of $f_x(a,b)$ or $f_y(a,b)$ does not vanish. Assume without loss of generality that $f_y(a,b) \neq 0$. The implicit function theorem (over $\mathbb{C}$) implies the existence of a holomorphic function $g_P(x)$ defined on an open set U_P containing a such that $f(x, g_P(x)) = 0$. This means that the curve $f(x,y) = 0$ around P is the graph of $y = g_P(x)$. Define the chart around P to be $(g_P^{-1}(U_P), \pi_P : (x,y) \mapsto x)$. Since at all points one of the derivatives does not vanish, by constructing the above chart for each curve we get an open covering of our curve. Of course, we may need to project to either the x-coordinate or the y-coordinate depending on the point.

To prove compatibility of the charts note that if P and Q both have charts projecting onto the x-coordinate, then $\pi_P \circ \pi_Q^{-1} = id_{U_P \cap U_Q}$ which is obviously holomorphic. The same holds if both charts project onto the y-coordinate. For the last case assume that the chart at P projects onto the x-coordinate and that the chart at Q projects onto the other and let $R \in g_P^{-1}(U_P) \cap g_Q^{-1}(U_Q)$ be in the intersection of both charts. On this set the curve is both of the form $y = g_P(x)$ and $x = g_Q(y)$, and so

$$\pi_Q \circ \pi_P^{-1}(z) = \pi_Q(z, g_P(z)) = g_P(z)$$
$$\pi_P \circ \pi_Q^{-1}(z) = \pi_P(g_Q(z), z) = g_Q(z)$$

which are both holomorphic.

Lastly, any irreducible algebraic curve over $\mathbb{C}$ is connected, this fact is not straightforward to prove, and certainly is not true over $\mathbb{R}$ (eg. $x^2 - y^2 = 1$, or $y^2 = x(x-1)(x-2)$). [4] in p.11 refers to Shafarevich's text on Algebraic Geometry for a proof. ∎

Theorem 8. *Every smooth projective plane alegbraic curve is a compact Riemann Surface.*

Proof. Let $F(x,y,z)$ be a polynomial satisfying the condition of smoothness described above. This implies (proof omitted) that $F(x,y,z)$ is irreducible (intuitively, if F were reducible then Bezout's theorem would imply the existence of singularities). Therefore, the affine parts of the projective curve C defined by $F = 0$ are given by

$$\begin{aligned} C_x : &\quad F(1,y,z) = 0 \\ C_y : &\quad F(x,1,z) = 0 \\ C_z : &\quad F(1,y,z) = 0 \end{aligned}$$

and are all smooth. By the previous lemma we get that C_x, C_y and C_z can all be given Riemann surface structure. What remains to be proved is that these charts are all compatible when viewed in the projective curve C in $\mathbb{CP}^2$.

Let P be a point in $U_x \cap U_y$. In U_x the coordinates are $(y/x, z/x)$ and in U_y they are $(x/y, z/y)$. In both affine parts we have charts projecting into either the first or the second factor (by the proof of the lemma above). Taking one of the cases, suppose that the chart ϕ_x in U_x projects onto the first factor and that the chart ϕ_y in U_y projects onto the second. Then $\phi_x([x : y : z]) = y/x$ and $\phi_y([x : y : z]) = z/y$. Now $\phi_x^{-1}(w) = [1 : w : h(w)]$ for some holomorphic function (by the proof of the lemma) and so

$$\phi_y \circ \phi_x^{-1}(w) = \phi_y([1 : w : h(w)]) = \frac{h(w)}{w}$$

which is holomorphic since $w \neq 0$ as $P \in U_x \cap U_y$. Similar arguments need to be given in a considerable amount of cases we omit (See [3] for all the possibilities).

For the statement about compactness take a point $P \notin C$. We can take an affine open set that contains P and the corresponding affine part of C, and transfer the picture to $\mathbb{C}^2$. The point P will not be in this affine part of the curve either, and this affine part of the curve is closed in $\mathbb{C}^2$ since it is the set where a polynomial vanishes. Therefore, there is a neighborhood of P in $\mathbb{C}^2$ which does not intersect C. This neighborhood gives an open neighborhood of P in $\mathbb{CP}^2$ which does not intersect C (remember the sets U_x, U_y and U_z are open in $\mathbb{RP}^2$). Therefore the set C is closed, and since $\mathbb{CP}^2$ is compact we conlude that C itself is compact. Note that this local argument is necessary since $F(x, y, z)$ is (regrettably) not even a function on $\mathbb{CP}^2$.

Connectedness follows from the fact that each affine part is connected and each part intersects the other two. ∎

We therefore get the following corollary.

Corollary 9. *Every smooth projective plane alegbraic curve is topologically a g-holed torus.*

The number g of holes is called the *genus of the smooth curve.* For irreducible singular curves we can actually associate a Rieman surface, often called the resolution of singularities of the curve, and the genus is defined as the genus of the resolution. See [2] Chapter 9 for references.

An important fact about the genus of a smooth curve is that it can also be defined algebraically allowing one to define the genus of a curve over arbitrary fields (the definition is considerably involved). The genus of a curve turns out to be a very important invariant even when one works over fields where there is no intuition of what it means. For example, an amazing result relating number theory, geometry and topolgy is the following: For a curve defined with coefficients over $\mathbb{Q}$, if the genus is strictly greater than 1 then the number of rational points on the curve is finite. This means, loosely speaking, that when one is wondering about the set of rational solutions to an equation with rational coefficients, the actual topological nature of the curve that this equation is defining is of enormous importance. In some sense, "Geometry governs Arithmetic". Of course, the relation between geometry and number theory goes far beyond this fact, but this is one of those instances where the connection is surprisingly strong.

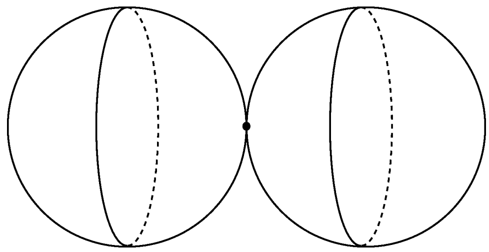

Figure 6: A misleading Picture of the intersection of two lines

3 Some insight coming from the Riemann surface structure

Our purpose in this section is to give some insight to some very well known facts from algebraic curves arising from what we know about their Riemann surface structure (at least for the smooth ones). None of the arguments which follow are proofs, but they nonetheless give tremendous insight to formulas that apparently seem to have no reason for being what they are in more general settings.

3.1 Lines in the Complex Projective Plane

As we saw, the complex projective line $\mathbb{CP}^1$ is diffeomorphic as a manifold to S^2. If we take any line in $\mathbb{CP}^2$ given by $ax + by + cz = 0$, then after a change of coordinates we can make the equation of the line become $z = 0$. Thus, any line is diffeomorphic to the set of points of the form $[x : y : 0]$ which is no other than $\mathbb{CP}^1$, i.e., *any line in* $\mathbb{CP}^2$ *is topologically a sphere.* Moreover, by extending Theorem 2 to $\mathbb{C}$, or by Bezout's theorem directly, we see that any two lines in $\mathbb{CP}^2$ will intersect at exactly one point.

Therefore, we see that trying to picture what is happening when two lines intersect in the four-dimensional complex projective plane $\mathbb{CP}^2$ is nothing but mind-boggling: The space is four dimensional (and moreover projective), lines are spheres, and each line (sphere) intersects any other line (sphere) in exactly one point. No matter which two lines one takes, the topological situation will be that of two spheres touching each other at one point. The first picture one can come up with in $\mathbb{R}^3$ (depicted in Figure 6) is misleading because we know that as far as our lines are distinct, their intersection will be *transversal* (they will not share the same tangent space at the point). A better picture of the situation would be the one in Figure 7 showing the transversality of the intersection, but then we have pictures non-smooth spheres and we know the spheres are smooth. Of course, what we are missing is one extra dimension where we can fit two smooth spheres touching at a point P and not having the same tangent plane at P!

We will nonetheless see (at least partially) the usefulness of being able

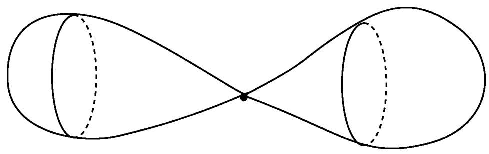

Figure 7: A better picture for the intersection of two lines

to picture the situation in the next section.

3.2 The genus in terms of the degree

A basic result from the theory of algebraic curves is that the genus of a smooth irreducible curve defined by a polynomial of degree d is given by the formula

$$g = \frac{(d-1)(d-2)}{2}.$$

It may (or may not) seem surprising that the genus of the curve is so intimately related to the degree of the defining polynomial. In this section we will give a geometric argument showing the plausibility of this formula and how one might guess it is true. The actual proof of the formula needs far more background than what is presented here.

We first show that if we fix a particular degree d, the set of projective curves defined by polynomials of degree d can be given the structure of a projective space by itself in which the equation of a curve corresponds to a point. To do this, note that the set of homogeneous polynomials of degree d is a vector space generated over $\mathbb{C}$ by the monomials of the form $x^i y^j z^k$ where $i+j+k=d$. With some combinatorics one can prove that there are exactly $(d+1)(d+2)/2$ monomials, and so this space has dimension $N=(d+1)(d+2)/2$. If we now fix an ordering of these monomials, we see that any homogeneous polynomial of degree d is uniquely specified by giving the coeficients $a_1, a_2, \ldots, a_N$ of each monomial. Considering now the curve that the homogeneous polynomial with coefficients $a_1, a_2, \ldots, a_N$ defines, we see that the polynomial $ka_1, ka_2, \ldots, ka_N$ defines the same curve for any $k \in \mathbb{C}^\times$, and so the space of degree d curves can be naturally identified with the projective space $\mathbb{CP}^{N-1}$ with coordinates $[a_1 : a_2 : \ldots : a_N]$.

Now, it may seem likely (this is one of the things we will not prove or make precise) that as one varies the coefficients continuously then the corresponding curves transform continuously. $\mathbb{CP}^{N-1}$ parametrizes all degree d curves in the sense described above. This will allow us to look at a particular curve of degree d and try to extract information about the nature of curves of degree d by seeing what we can do to the topology of the curve with small perturbations. Some of the curves will be singular, but the definition of singular curve shows that they all lie in a "small set" (I am uncertain of

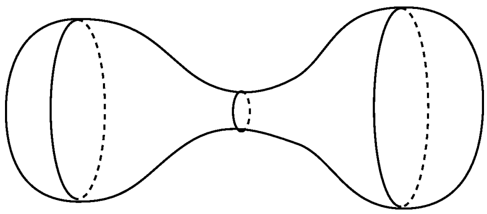

Figure 8: A smooth curve of degree 2.

whether the set of points defining singular curves defines an algebraic set, but it doesn't seem unlikely).

Let us look at the case of degree 2 and 3 in some detail. For curves of degree 2 we may look at the curve $xy = 0$. We consider this curve because it is the union of two lines and we already know what the topological situation is in this case. Specifically, the curve $xy = 0$ looks topologically like two spheres touching at one point transversally as in Figure 7. This curve is obviously singular, but considering the effect of a small perturbation of the coefficients one expects to be able to obtain a non-singular curve, and moreover, that the effect of the perturbation is that it replaces the point of contact of the spheres by a smooth "neck" connecting them as in Figure 8. From this one would guess that curves of degree 2 have genus zero, and this is in fact the case. This is our formula $g = (d-1)(d-2)/2$ for the case $d = 1$.

For curves of degree 3 we may also take the particularly simple curve $xyz = 0$. In this case we have three lines, each intersecting the other two transversally. In terms of the surfaces, we have 3 spheres, each intersecting the other two transversally at one point. A picture of this can be seen in Figure 9. This curve has three singularities, and perturbations of the coeficients will smoothen out these by replacing them by necks connecting the spheres. This gives a torus, whose genus is 1. This is in fact the case: any nonsingular conic has genus 1.

Note, however, that in these pictures one does not see why the resolution of singurarities of a singular cubic is a sphere, or why the genus drops when singularities are present. These pictures do not serve concretely to infer results about curves, but they do give us the genus of a member of the family. What we tried to do above was obtaining the highest genus possible.

For higher degrees one may use similar configurations of lines. In general, the curve will not be defined by a monomial because we only have three variables, but any union of lines in "general" position will do (by general position we mean that no three lines intersect at the same point). For example, for degree 4 we get four lines and so we are led to consider an arrangement of spheres in space located in the vertices of a tetrahedron. Smoothening the intersections gives a surface of genus 3 (See Figure 10). Again, $(4-1)(4-2)/2$ gives 3. The situation regarding the surfaces will

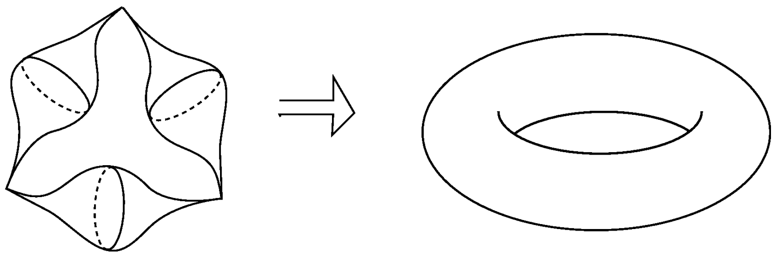

Figure 9: A singular Cubic.

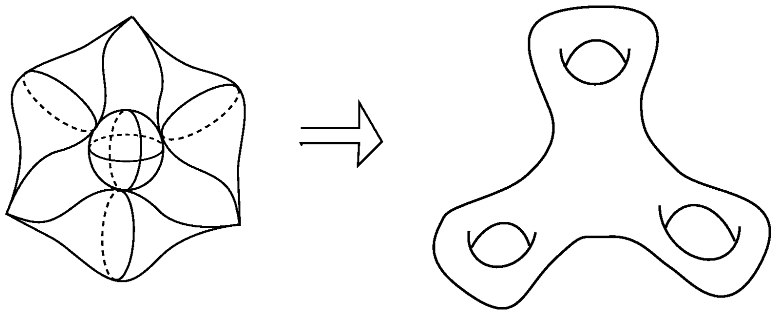

Figure 10: Smooth curves of degree 4 have genus 3.

be that of d spheres intersecting at $d(d-1)/2$ points (one for each set of two spheres). One can prove that if one replaces the points of intersection by smooth necks that one obtains a surface with $(d-1)(d-2)/2$ holes (see [3] for the argument). This gives the formula stated at the beginning of the section.

In the realm of algebraic geometry the notion of equivalence is more strict than the topological one we have been discussing. One only considers *rational* maps (maps given by rational functions) between curves, and equivalence of curves is characterized by the existence of *birrational* maps (rational maps with rational inverses). It is a very surprising fact that these topological results carry all the way to this more restrictive setting. The topological equivalence of curves of degree with lines (i.e., spheres) translates and generalizes to the fact that every conic is birrationally equivalent to a line. This in turn implies that every conic is parametrizable by rational funtions, and all is intimately related to these topological pictures we have been considering. The algebraic genus of a curve is a birrational invariant, and as was mentioned earlier, is intimately related to the arithmetic of the curve.

4 What comes next

Here is an annotated list of possible topics to follow up this term paper.

The Riemann Hurwitz Formula. This formula relates the genus of two compact Riemann surfaces when there exists a holomorphic map between them. The formula involves the genus of both surfaces, and the number of sheets and order of the branching points of the map between the surfaces. With it, one can prove the degree genus formula for smooth curves by projecting the curve to the complex projective plane and studying the branching points. Contents could include:

- Triangulations of surfaces
- The Euler characteristic and why it is well defined.
- Definition of the genus of a surface.
- Branched coverings
- Proof of the degree-genus formula using this material.
- An explanation of the example found in Raoul Bott's article: On the shape of a curve.

References : Fulton, Miranda, Brieskorn, Kirwan, Fischer, Bott's article.

The genus of singular curves. Resolution of singularities in the various ways it is possible, showing that the genus thus defined is independent of what method is used.

- Construction of a Rieman surface S and a map between S and the set of nonsingular points of the curve which is biholomorphic. (Fischer's book Chapter 9 is a good place to start).
- Defining the genus of the curve as the genus of S, and why this is well defined.
- Blowing up points: Over arbitrary fields, and the special case when the field is $\mathbb{C}$ where the topological situation can be studied (Brieskorn's σ-process).
- As much examples as possible of both things.

References: Brieskorn, Fischer, Kirwan, Miranda.

Topology of Conics and Cubics Some analysis of curves of low degrees, studying how singularities behave, how they appear and the shape of the singular sets in the parameter space. (Brieskorn has at least two degrees worked out).

- The surface of the singularities of families of conics.
- Elliptic curves from topology: Brieskorn pg. 313.

· How the torus of cubics changes when singularities appear.
· The y^2= quartic...why is it a torus?

References: Brieskorn, Miranda CH III p.57.

Intersection of Curves

The Topology of Bezout's Theorem

References: Brieskorn.

The topology of singularities of complex curves Brieskorn contains lots of material about the topological nature of singularities of curves. This includes associating knots to singularities!

Hopf Fibrations Study the maps $S^{2n+1}/S^1 \to \mathbb{RP}^n$, in particular $S^3/S^1 \to S^2$. They are briefly mentioned in Brieskorn p. 138.

References

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