Lines on cubic surfaces

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Cubic curves

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- Example: If we want the polynomials to vanish at the point $p = (1, 1)$, then we must have

$$a_1 + a_2 + a_3 + \ldots + a_{10} = 0$$

The construction

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- Define the map

$$\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x, y) & \mapsto & \left(\frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0}\right) \end{array}$$

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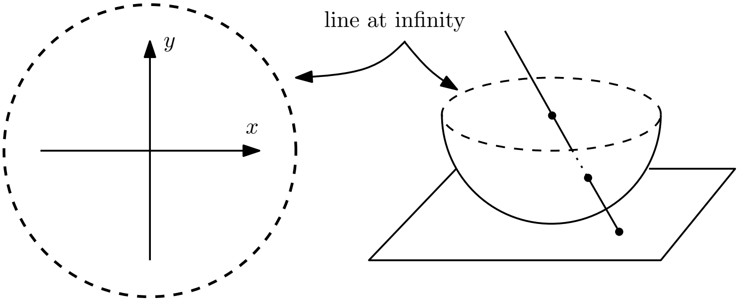
All these infinities correspond to directions.



The points where the map is not defined

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- At the points where $f_0 = 0$ but any other $f_i \neq 0$, the image is trailing off to infinity.
- At the points where all the f_i vanish (i.e., at $p_1, \ldots, p_6$) something entirely different is happening.

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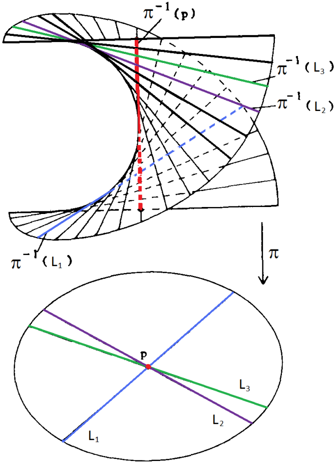
which is now defined and continuous on all points of the x-axis except at the origin.

Blowing up points

- Given any smooth algebraic surface, replace a point by a line (the Exceptional Line): a copy of $\mathbb{P}^1$.
- Each point on this line corresponds a tangent direction at the point.



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which is now defined at every point!

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- This map is defined everywhere.
- It is an isomorphism onto its image.
- The image is a smooth cubic surface in $\mathbb{P}^3$.

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Theorem

Any smooth cubic surface in $\mathbb{P}^3$ *contains exactly 27 lines.*

Cubic Surfaces and blow up's

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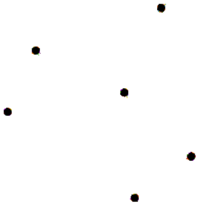

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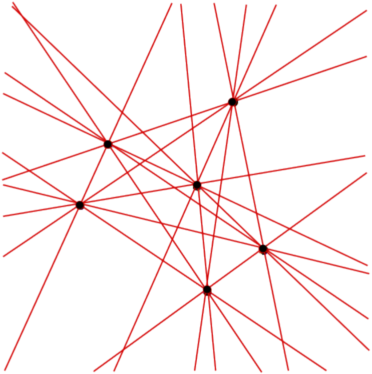

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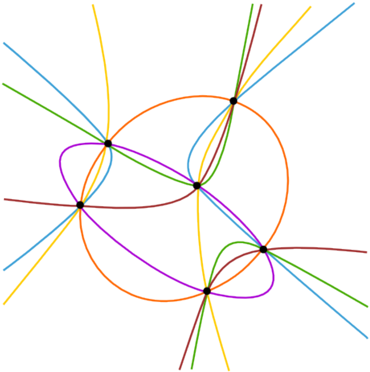
6: The strict transform of the 6 (unique) conics through five of the six points.



The Clebsch Diagonal Cubic

$$x^3 + y^3 + z^3 + w^3 - (x + y + z + w)^3 = 0$$

Image Credits

The Clebsh Cubic (blue):

By Oliver Labs, from his webapge *The Cubic Surface Homepage* at http://www.cubics.algebraicsurface.net/