The Math behind Escher's *Print Gallery*

Enrique Acosta

Department of Mathematics
University of Arizona

September 2009

What Escher wanted to do

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- A “self referential” picture


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- As you go around the picture the scale increases so that when you return to where you started, you are now inside the self reference.

A town with a gallery and a guy looking at a picture of the town with the gallery and him looking at the picture ...
but he is both the guy looking at the picture and the guy in the picture!

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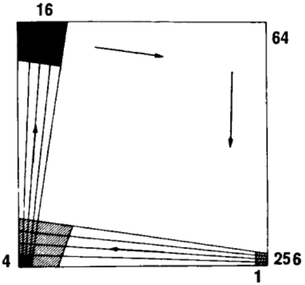
- As you go around the picture the scale increases so that when you return to where you started, you are now inside the self reference.
- As you go along each side the scale increases by a factor of 4.



How he did it

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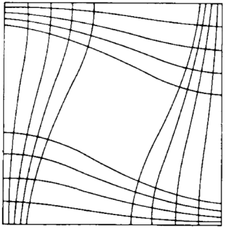
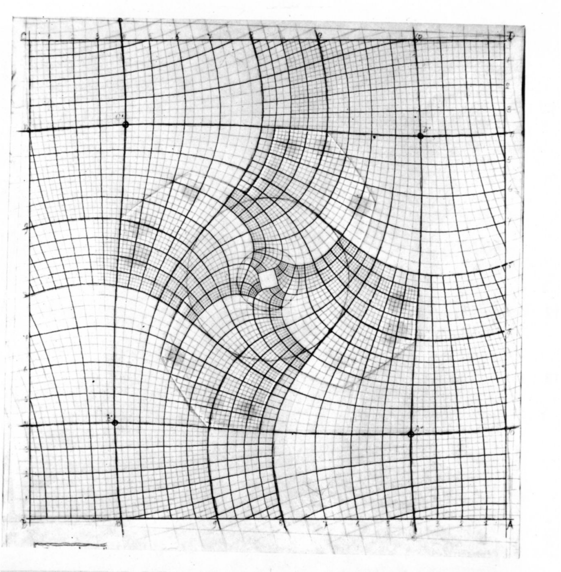
- Use curved grid to preserve angles.



Escher's Grid



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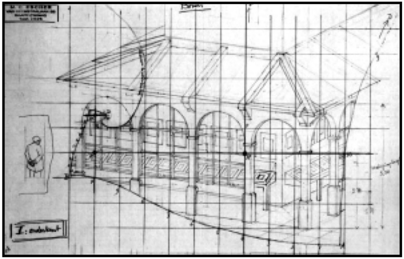

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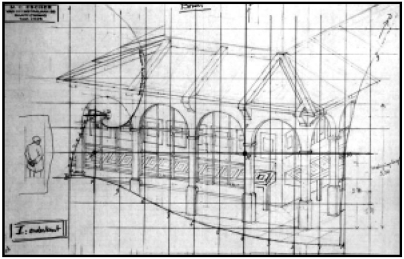
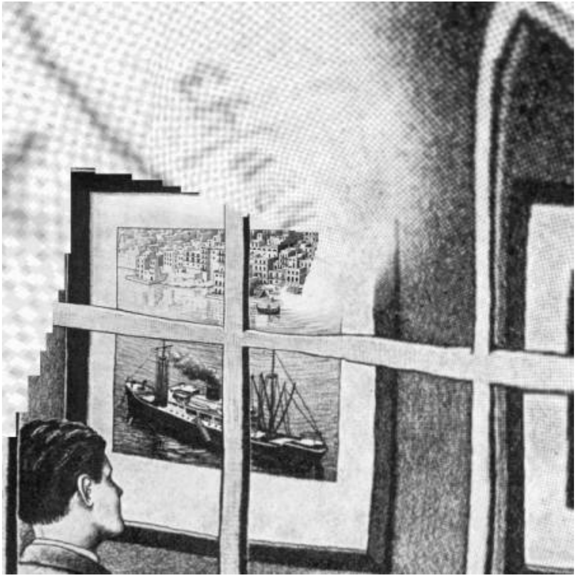
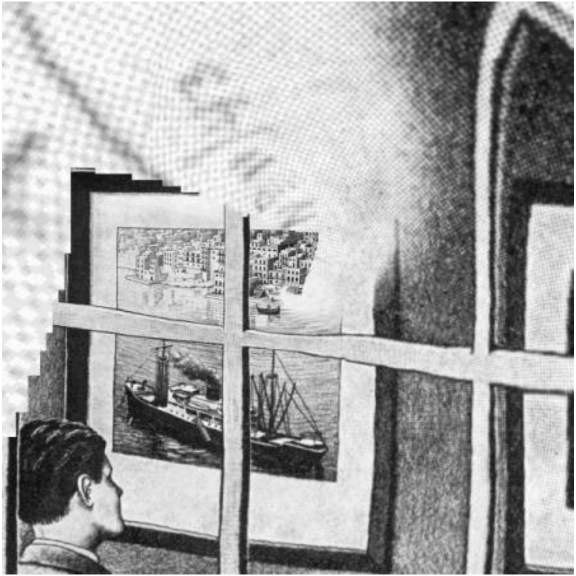
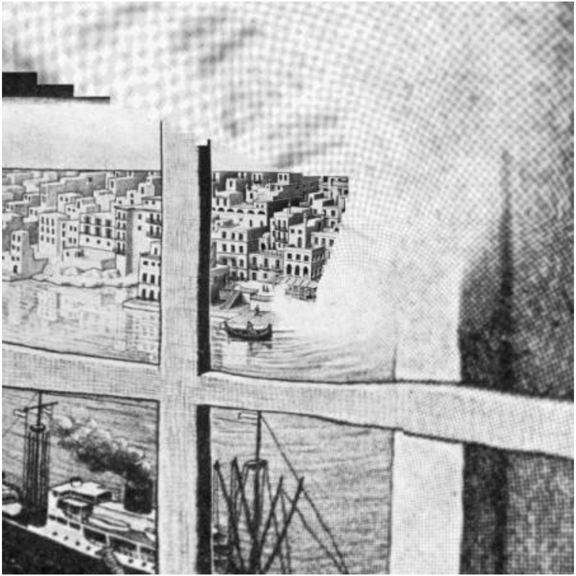
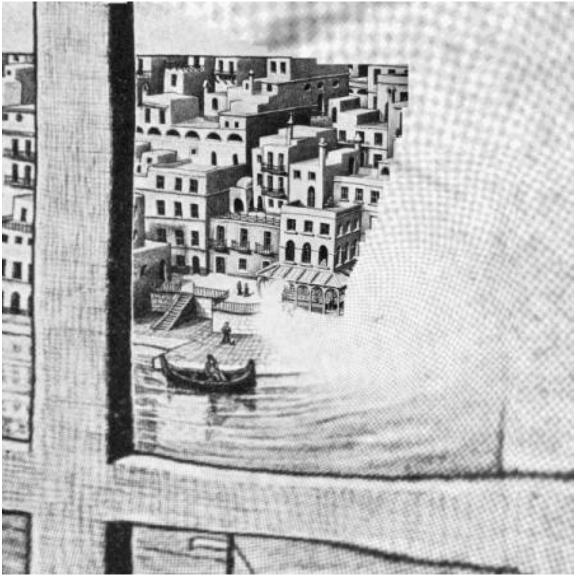


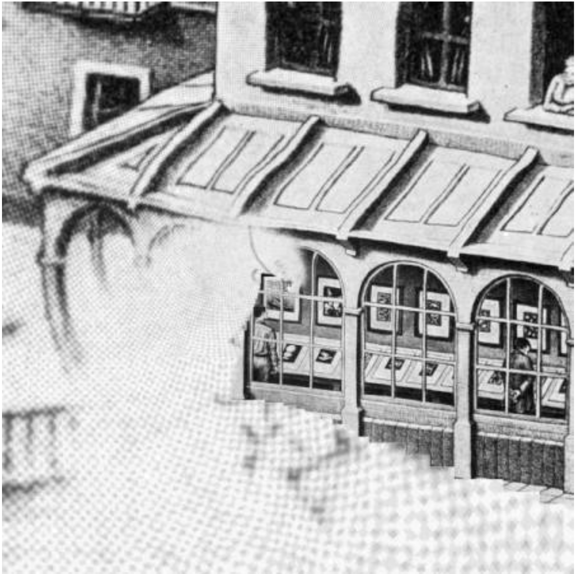
- Using his grid we can "unwind" his final picture to see the straight one.

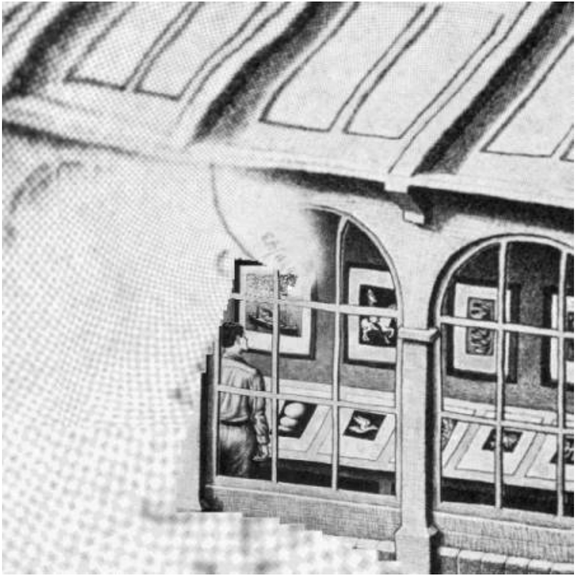


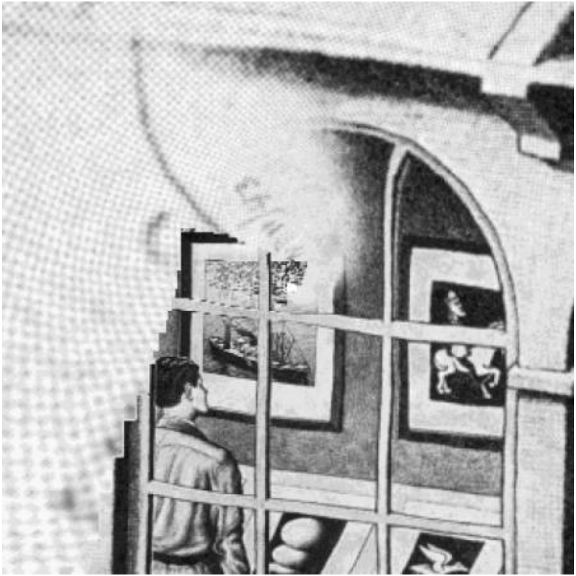


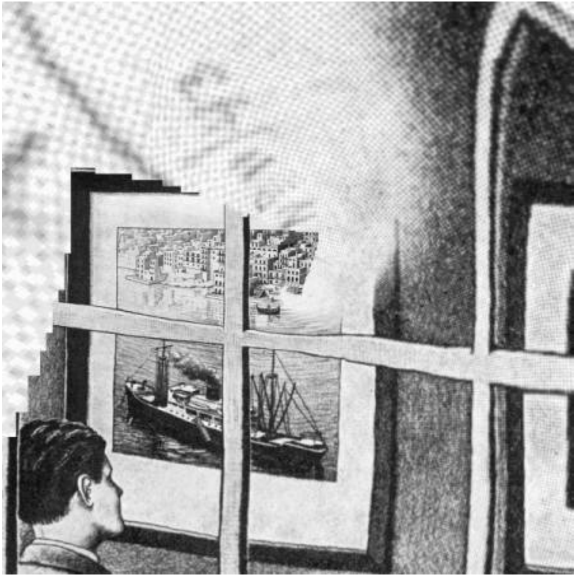












The Math

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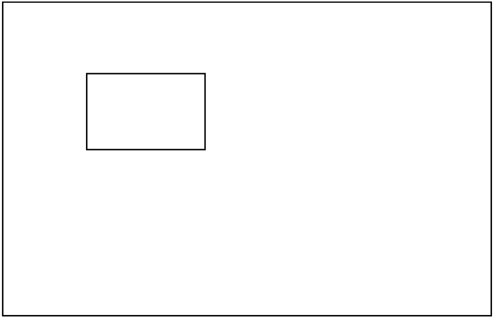
Every image that contains a copy of itself has a center.



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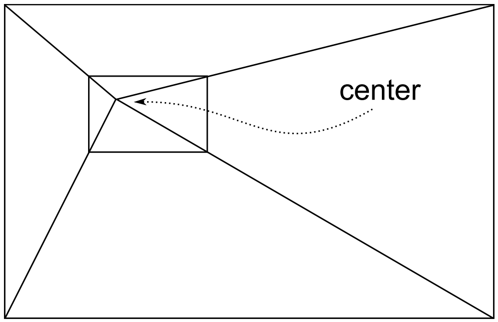

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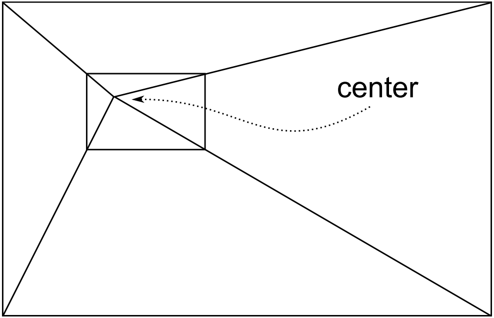
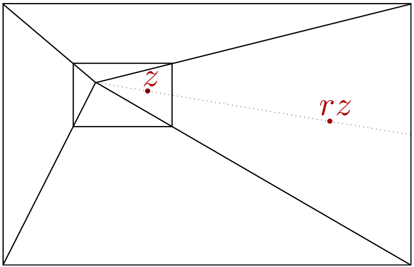


Now place the picture in the complex plane with the center at the origin.

The Math



- The fact that the picture contains a copy of itself just means that it is invariant under multiplication by a scalar r, which we will call the period.

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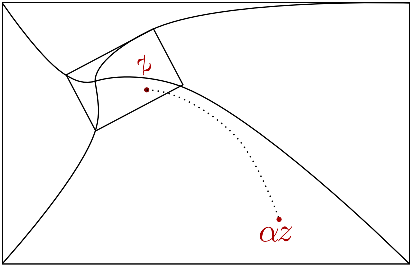

- The fact that the picture contains a copy of itself just means that it is invariant under multiplication by a scalar r, which we will call the period.
- Escher's twisted picture SHOULD have a *complex period*. That is, it should be a picture that is invariant under both a rotation and a scaling.

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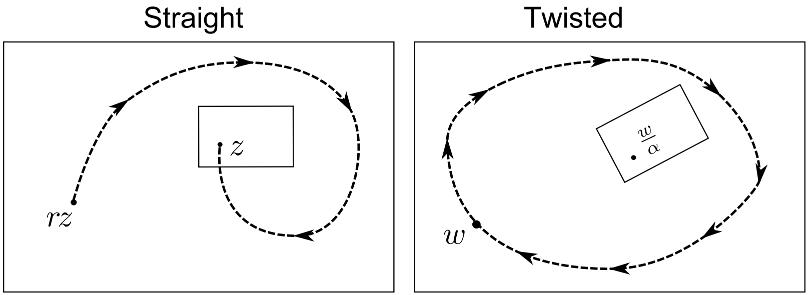
  - The transformed picture has complex period.
  - A loop in the transformed picture around the center corresponds to a path around the center of the original picture going from a point z to rz (r is the real period).



Images with complex period

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- A black and white image with a complex period δ can be though of as a map

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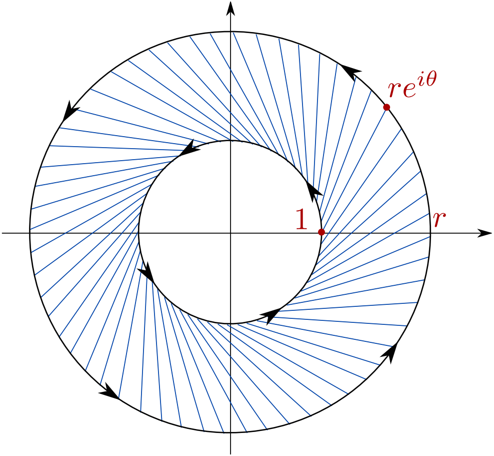
- If we drop the origin we can actually think of this f as a function

$$f : \mathbb{C}^*/\langle\delta\rangle \to \{\text{black, white}\}$$

where $\langle\delta\rangle = \{\delta^n \mid n \in \mathbb{Z}\}$ is the subgroup of $\mathbb{C}^*$ generated by δ.

¿ What is $\mathbb{C}^*/\langle\delta\rangle$ for $|\delta| > 1$?

Let $\delta = re^{i\theta}$. We can picture $\mathbb{C}^*/\langle\delta\rangle$ like below:



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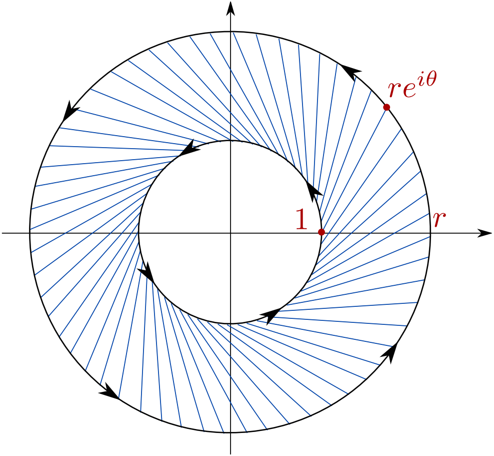

It is a Torus!

Coverings, Fundamental Groups

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- The isomorphism $\mathbb{C}/L_\delta \cong \mathbb{C}^*/\langle\delta\rangle$ is actually an *isomorphism* of Riemann surfaces (i.e. it is biholomorphic) and gives us another way to picture the torus $\mathbb{C}^*/\langle\delta\rangle$.

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- If $|\delta| \neq 1$, we can think of L_δ as the fundamental group of the torus.

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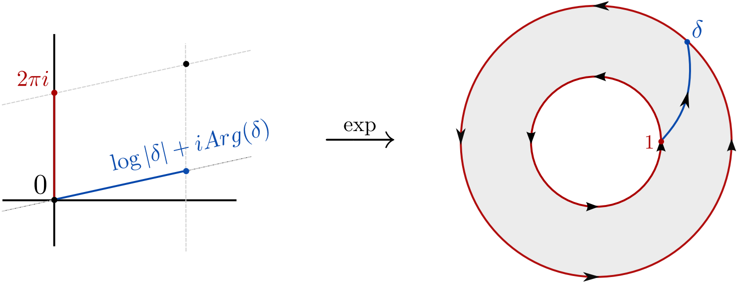
- Every element of L_δ corresponds to a unique loop of $\mathbb{C}^*/\langle\delta\rangle$.
- This loop is the image under the exponential map of any path starting at 0 and ending at the element of L_δ.
- Alternatively, L_δ are the endpoints of the lifts of loops in $\pi_1(\mathbb{C}^*/\langle\delta\rangle, 1)$.



Back to Escher's picture

- Start with a picture with real period r,

$$f : \mathbb{C}^*/\langle r \rangle \to \{\text{black, white}\}$$

- Transform it with a map $\mathbb{C}^* \to \mathbb{C}^*$ so that:
  - The map is conformal.
  - The map sends 1 to 1.
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  - The process can be reversed.
- If this happens then we get an isomorphism (biholomorphic) $\mathbb{C}^*/\langle r \rangle \to \mathbb{C}^*/\langle \delta \rangle$ between Riemann surfaces sending 1 to 1.

Back to Escher's picture

- By the theory of covering spaces this map lifts to a map

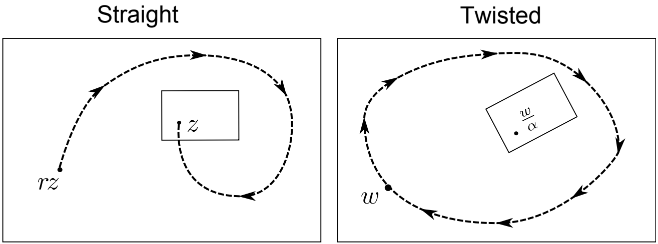
$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ \mathbb{C}^*/\langle r \rangle & \longrightarrow & \mathbb{C}^*/\langle \delta \rangle \end{array}$$

- The top map is holomorphic and we may assume that it sends 0 to 0.
- This map induces isomorphism $\mathbb{C}/L_r \to \mathbb{C}/L_\delta$ between Riemann surfaces.
- The theory of complex tori as Riemann surfaces (or complex elliptic curves) implies that the map $\mathbb{C} \to \mathbb{C}$ is multiplication by a scalar α which satisfies

$$\alpha L_r = L_\delta.$$

Loop Correspondence

- The loop correspondence



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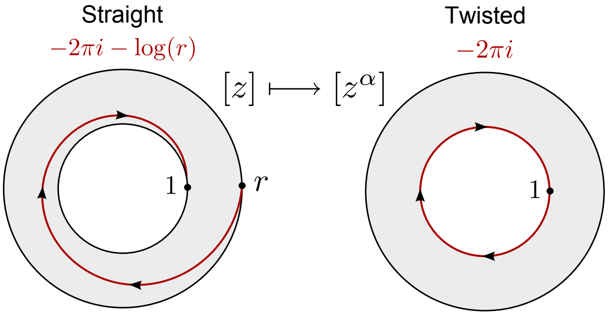

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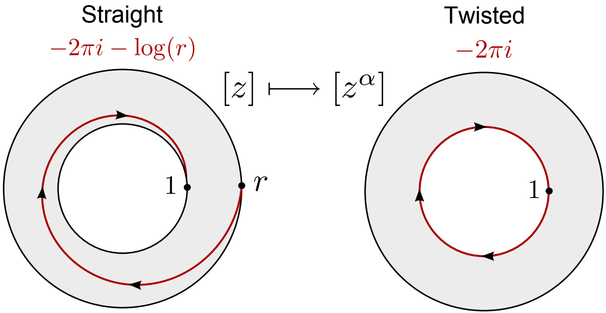

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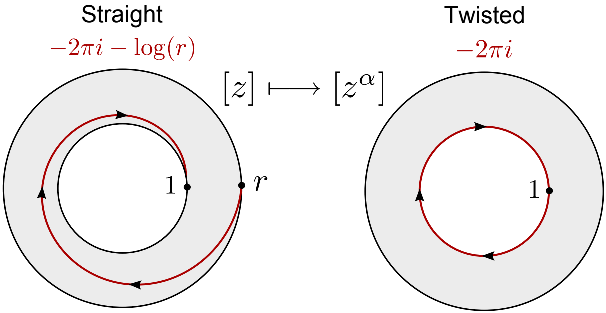

implies that multiplication by α sends the element $2\pi i + \log(r)$ to the element $2\pi i$.

- So

$$\alpha = \frac{2\pi i}{2\pi i + \log(r)}$$

Loop Correspondence

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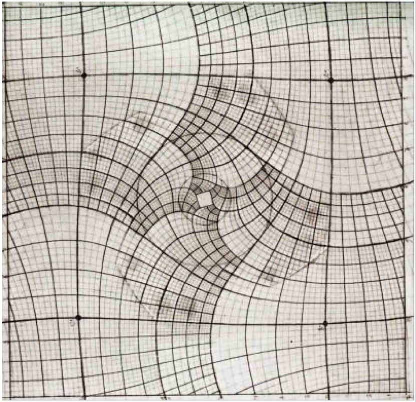
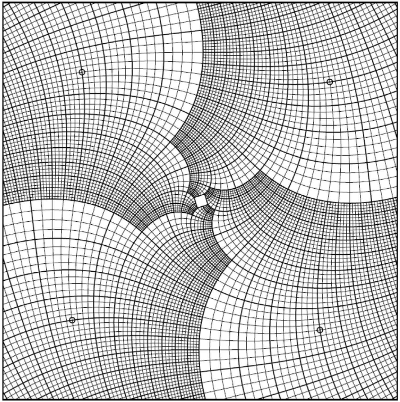
- Finally, since we know α, and $\alpha L_r = L_\delta$ allows us to find δ.
- So, starting with r and the loop correspondence that Escher wanted, we can find α and δ. So we can produce Escher's picture mathematically.

The mathematically precise grid





The mathematically precise picture

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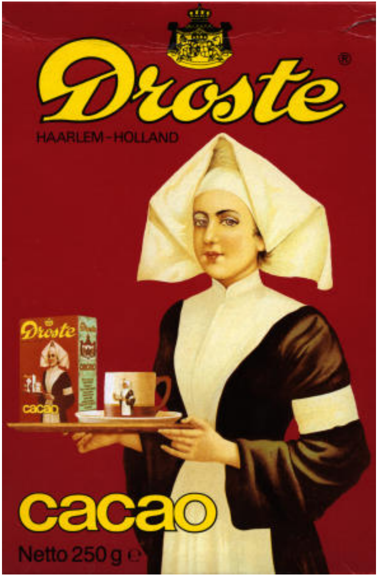
VIDEO

Credits

- Hendrik Lenstra (Elliptic curve factorization) started to wonder about this on a plane when he found the picture on a magazine.
- Him and Bart de Smit wrote the article *The Mathematical Structure of Escher's Print Gallery* which was published in the Notices of the AMS on April 2003.
- They also created a webpage with lots of images and videos. http://escherdroste.math.leidenuniv.nl/
- Google "Escher Droste" to see what people have done with this.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Variations

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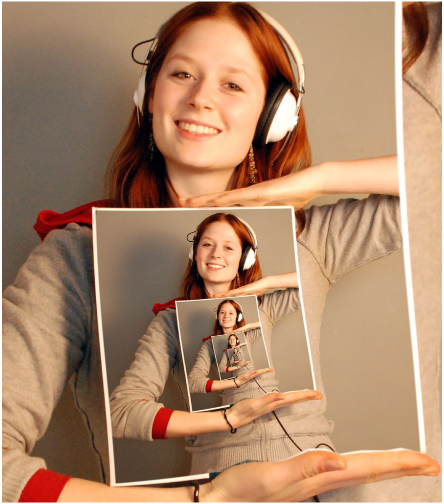

Form Flickr's Escher Droste Print Gallery group (1540 images)
trufflepig droste copy by manyone1

VIDEO