$\overline{M_{0,n}}$
Part I

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(Fine) Moduli Space of Curves

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Family of Curves

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Example

If P says the curve has only one singularity, then a family $X \to B$ should come with a morphism $s : B \to X$ (a section) that tell us which is the singular point on the fiber.

Equivalence of Families of P-Curves

We say that $X \to B$, $Y \to B$ are equivalent families of P-curves if there is an isomorphism $X \to Y$ making the diagram commute

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

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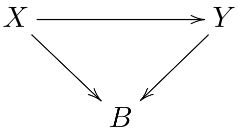
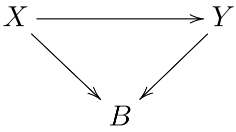


and that preserves all the extra structure that P imposes (this needs to be specified precisely and depends on P).

Example

If P says there is a unique singular point, then it is natural to require that the map $X \to Y$ should be taking the singular points in the fiber to the corresponding singular points in the other fiber.



Equivalence of P-curves

- The notion of equivalence of families includes the notion of equivalence of P-curves since we can take $B = \{\bullet\}$.
- We call this a P-isomorphism.

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  - This functor is isomorphic to $\mathrm{Hom}(-, \mathcal{M}_P)$.

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As an example, take a curve C in $\mathcal{M}_P$.

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- There is in general no good reason to expect to be able to glue all these curves into a surface, but in this case the properties of $\mathcal{M}_P$ imply the existence of this surface!
- The simple inclusion $C \hookrightarrow \mathcal{M}_P$ gives automatically a family $X \to C$ with X_p (fiber above p) equal to the P-curve that it corresponds to.

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The Universal Family

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- Moreover, since any family $X \to B$ gives a map $B \to \mathcal{M}_P$, then functorial arguments show that $X \to B$ is the pullback of the family $U \to \mathcal{M}_P$.
- This is why U is called the universal family!

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- Two families $X_1 \to B$ and $X_2 \to B$ are said to be equivalent if there is an isomorphism $X_1 \to X_2$ which preserves the fibers together with their identification as lines in k^{n+1} coming from the structure map.
- The universal family in this case is $U = \{[l] \times x \mid x \in l\} \subset \mathbb{P}^n \times k^{n+1}$.

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- Family: $X \to B$ with X_b a genus zero curve for all $b \in B$ and n disjoint sections $s_1, \ldots, s_n : B \to X$ which give you the distinct n marked points on X_b.
- Equivalence of families is given by isomorphisms $X_1 \to X_2$ that preserve fibers and send sections to sections.

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- There is a (fine) moduli space for $n \geq 3$! We call it $M_{0,n}$.

$\mathbf{M_{0,3}}$

$M_{0,3} = \{\bullet\}$ since there is only one equivalence class of smooth genus zero curves up to 3-isomorphism.

$\mathbf{M_{0,4}}$

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- We have
$$M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$$
(this needs to be checked!)

Example (Renzo's)

- Consider the family of 4-curves $C_t = (\mathbb{P}^1, 0, 1, \infty, t)$ over $\mathbb{A}^1 - \{0, 1\}$ (it does not extend to a family of 4-curves over 0 or 1 because two points would agree).

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- Explicitly, the family given by

$$\begin{array}{rcl} \mathbb{P}^1 \times (\mathbb{A}^1 - \{0,1\}) & \to & \mathbb{A}^1 - \{0,1\} \\ p \times t & \mapsto & t \end{array}$$

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  with the three constant sections $0, 1, \infty$ and the section $t \mapsto t \times t$.
- Since $M_{0,4}$ is a fine moduli space, then this family gives a morphism

$$\mathbb{A}^1 - \{0,1\} \to M_{0,4}$$

$C_t = (\mathbb{P}^1, 0, 1, \infty, t)$

- Now $D_t = (0, t^{-1}, \infty, 1)$ is another family over $\mathbb{A}^1 - \{0, 1\}$ and so gives another morphism

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- This implies that the families C_t and D_t for $t \neq 0$ give the same morphism from $\mathbb{A}^1_t - \{0, 1\}$ to $M_{0,4}$.

$\mathbf{M_{0,n}}$ for $n \geq 3$

- In general, sending the first three points in $p_1, \ldots, p_n$ to $0, 1, \infty$ gives the tuple $p'_4, \ldots, p'_n \in \mathbb{P}^1 - \{0, 1, \infty\}$ which determines the curve up to n-isomorphism.

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- One can prove that

$$M_{0,n} = (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \{\text{diagonals}\}$$

  where one identifies the n-curve with the p'_i (none of the p'_i can agree!)

The Universal family over $M_{0,n}$

The universal family is given by

$$\begin{array}{c} U_n = M_{0,n} \times \mathbb{P}^1 \\ \downarrow \\ M_{0,n} \end{array}$$

where the sections come from the constant sections $0, 1, \infty$, and the sections s_i is given by

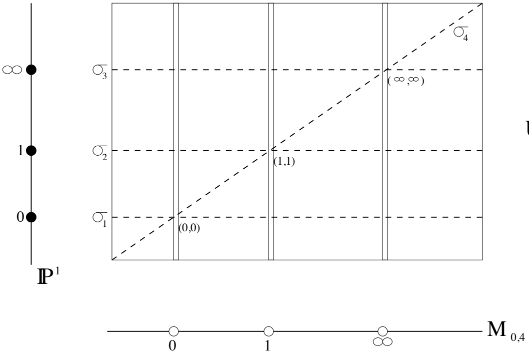
$$\begin{array}{rcl} s_i : M_{0,n} & \to & U_n \\ (p'_4, \ldots, p'_n) & \times & (p'_4, \ldots, p'_n) \times p'_i \end{array}$$

for $i = 4, \ldots, n$.

The Universal family over $M_{0,n}$

Example $n = 4$

s_4 is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ minus the points $0 \times 0, 1 \times 1, \infty \times \infty$ which lie over the points in $\mathbb{P}^1 - M_{0,n}$.



(picture taken from Renzo's notes)

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- We first analyze the case $n = 4$ an see some motivation for the definition.

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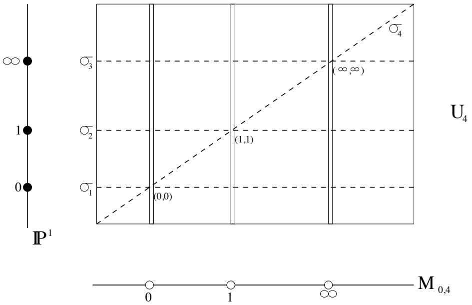
Take $M_{0,4}$ and its universal family U_4



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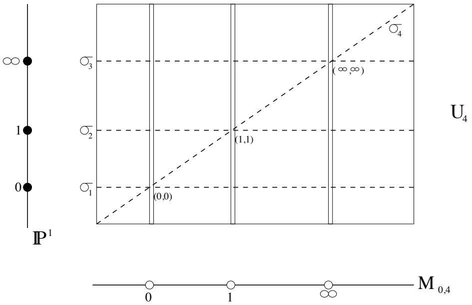
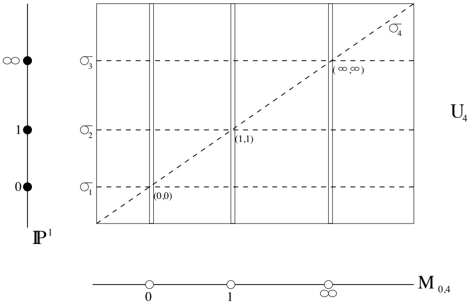


- The family is not defined at $0, 1, \infty$ because p_4 can't be any of these.
- One may think that the answer is to enlarge $M_{0,4}$ to $\mathbb{P}^1$, and let the three extra curves it parametrizes be the ones above, where $p_1 = p_4$ above 0, $p_2 = p_4$ above 1 and $p_3 = p_4$ above ∞.



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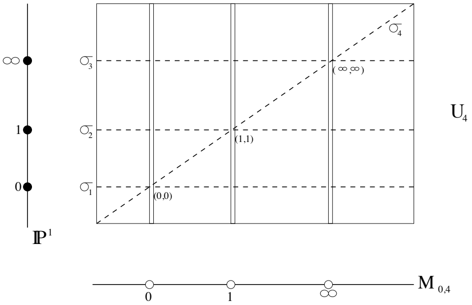

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- It also does not work!

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- These configurations are not equivalent up to 4-isomorphism, and so should be considered as distinct points in our compactification of $M_{0,4}$.
- Thus, our map $\mathbb{A}^1_t - \{0, 1\} \to M_{0,4} \subset \mathbb{P}^1$ would need to extend in two different ways at $t = 0$!

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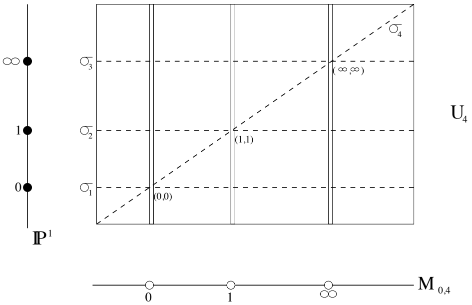
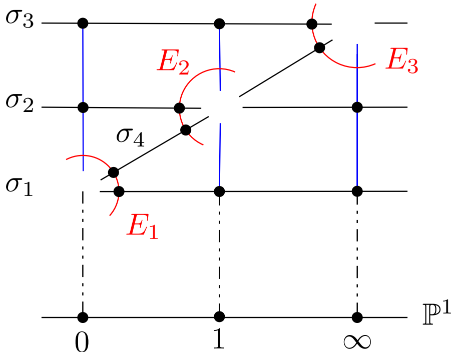
- For $t = 0$, C_0 has $p_1 = p_4$ whereas D_0 has $p_2 = p_3$.
- These configurations are not equivalent up to 4-isomorphism, and so should be considered as distinct points in our compactification of $M_{0,4}$.
- Thus, our map $\mathbb{A}^1_t - \{0, 1\} \to M_{0,4} \subset \mathbb{P}^1$ would need to extend in two different ways at $t = 0$!
- This at least hints at the fact that, whatever our space should be, it should identify the situations $p_1 \to p_4$ and $p_2 \to p_3$.

$\overline{M_{0,4}}$: The good way to compactify $M_{0,4}$

We have $U_4 \subset M_{0,4} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$

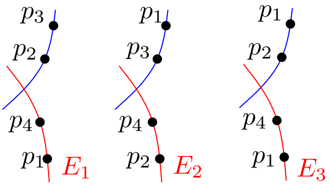


- Blow-up $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $0 \times 0, 1 \times 1, \infty \times \infty$.
- This separates the sections and so we get a family of curves that extends to all of $\mathbb{P}^1$ where all the fibers now have 4 distinct marked points.



σ_3
σ_2
σ_1
σ_4
E_1
E_2
E_3
$\mathbb{P}^1$
0
1
∞

$\overline{M_{0,4}} = \mathbb{P}^1$



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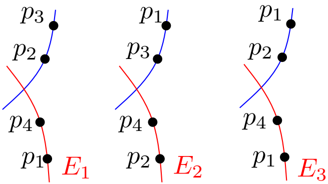

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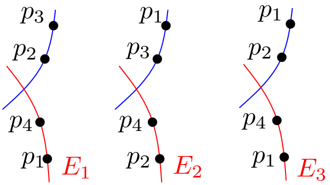

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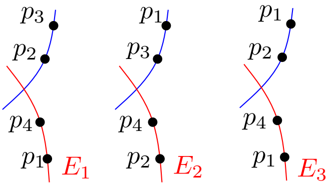

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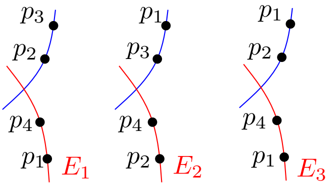

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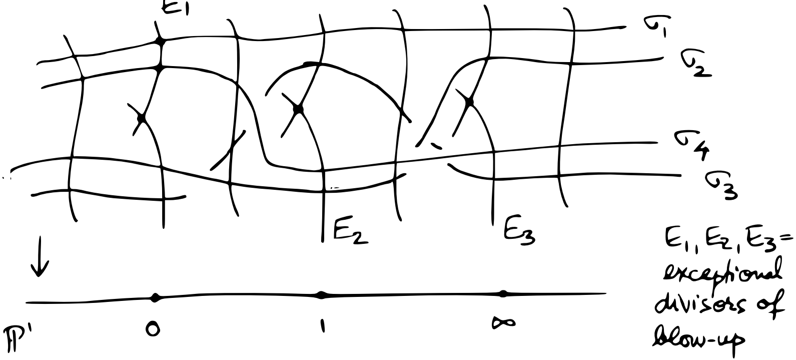
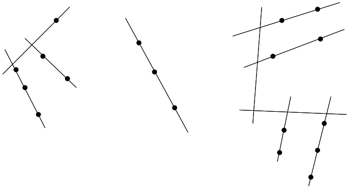
- These 3 new curves have no nontrivial 4-automorphisms.
- Instead of letting the points collide, the space $\overline{M_{0,4}}$ adds one more $\mathbb{P}^1$ where it puts the points that tried to collide. This is the way it stores the information that the collided.

Ana-Maria's picture of the situation:



Stable genus zero n-pointed curves (stable n-curves)

A stable genus zero n-curve is a curve with n marked points that has the following properties:



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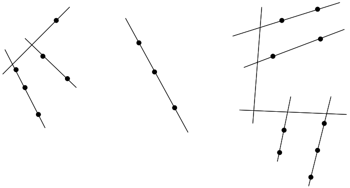

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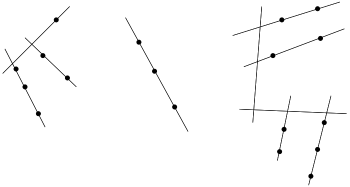

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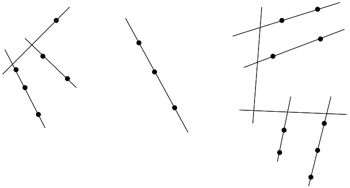

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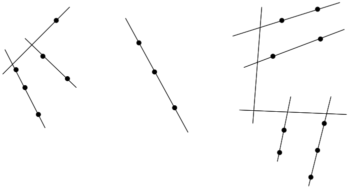

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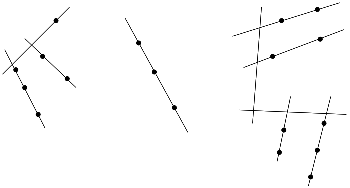

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- Each component has at least 3 special (singular or marked) points, and no marked point is singular.

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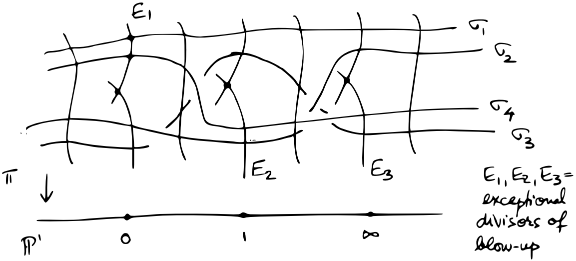
Some History

It was first constructed by Knudsen in 1983. One of the highlights of modern algebraic geometry.

Example: The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$!



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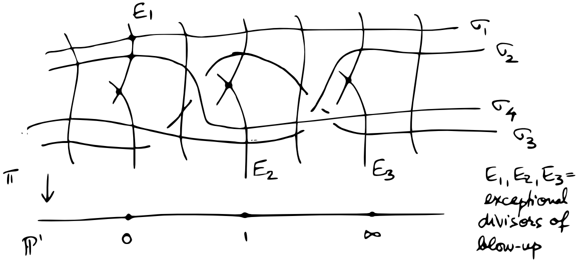

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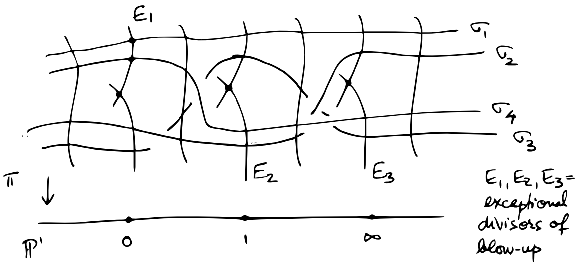

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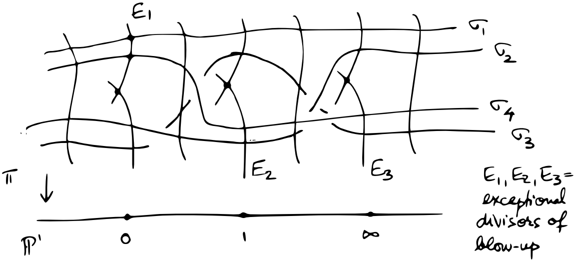

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- Otherwise we need to *stabilize* the curve.

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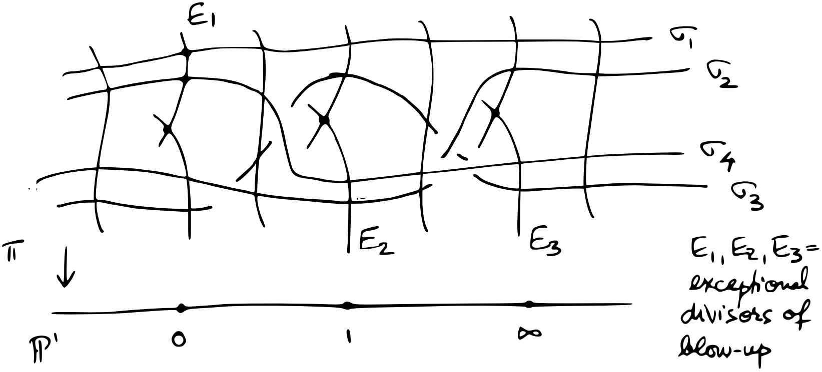
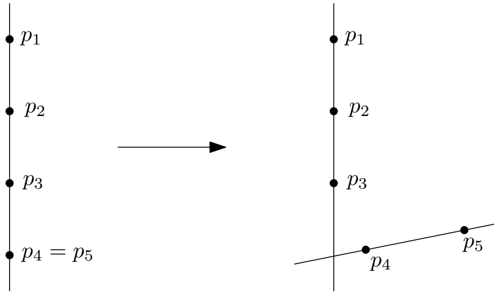


Stabilization

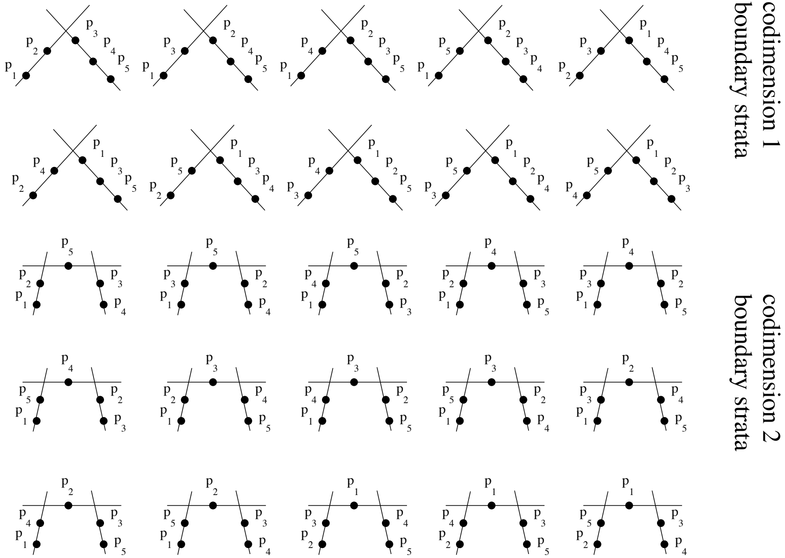
- If p is not in one of the sections, but it is a singular point of one of the special ones, then replace the singular point by a $\mathbb{P}^1$ and place $p = p_5$ in this new $\mathbb{P}^1$.

Stabilization

- If p is in one of the sections, say σ_4 then take the fiber and add a $\mathbb{P}^1$ at $\sigma_4(q)$ and put p_4 and p_5 on this new $\mathbb{P}^1$ (this is unique up to isomorphism).



This takes care of all isomorphism classes of 5-curves!



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References

- Kapranov:
  - "Veronese curves and Grothendieck-Knudsen moduli space M(0,n)", Journal of Algebraic Geometry, 1993.
  - "Chow quotient of Grassmannians I", Adv. Soviet Math.
- Ana-Maria Castravet: Course, and course notes available online, "Topics in Geometry and Topology (Moduli of curves)", Spring 2010.
- Renzo Cavalieri: Course notes available online, "Math 676 Topics: Moduli Spaces", Fall 2010.