$\overline{M_{0,n}}$
Part II

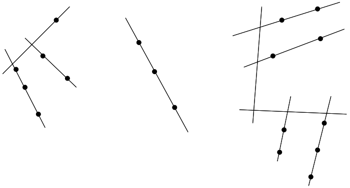
Enrique Acosta

Department of Mathematics
University of Arizona

November 2011

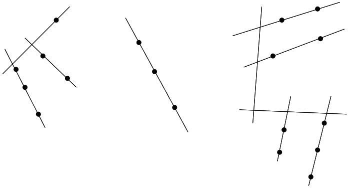
Stable genus zero n-pointed curves (stable n-curves)

A stable genus zero n-curve is a curve with n marked points that has the following properties:



Stable genus zero n-pointed curves (stable n-curves)

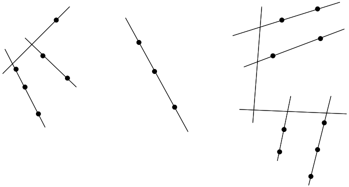
A stable genus zero n-curve is a curve with n marked points that has the following properties:



- It is connected.

Stable genus zero n-pointed curves (stable n-curves)

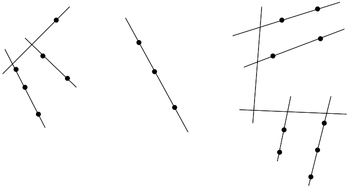
A stable genus zero n-curve is a curve with n marked points that has the following properties:



- It is connected.
- Each irreducible component is isomorphic to $\mathbb{P}^1$.

Stable genus zero n-pointed curves (stable n-curves)

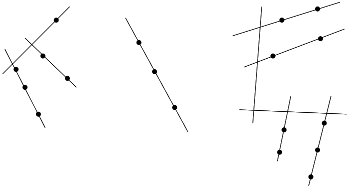
A stable genus zero n-curve is a curve with n marked points that has the following properties:



- It is connected.
- Each irreducible component is isomorphic to $\mathbb{P}^1$.
- If two $\mathbb{P}^1$'s intersect, then they do so transversally.

Stable genus zero n-pointed curves (stable n-curves)

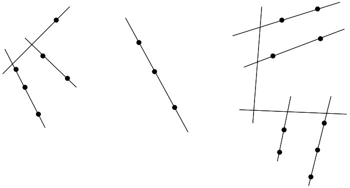
A stable genus zero n-curve is a curve with n marked points that has the following properties:



- It is connected.
- Each irreducible component is isomorphic to $\mathbb{P}^1$.
- If two $\mathbb{P}^1$'s intersect, then they do so transversally.
- The curve has (arithmetic) genus zero. This implies that there are no closed circuits.

Stable genus zero n-pointed curves (stable n-curves)

A stable genus zero n-curve is a curve with n marked points that has the following properties:



- It is connected.
- Each irreducible component is isomorphic to $\mathbb{P}^1$.
- If two $\mathbb{P}^1$'s intersect, then they do so transversally.
- The curve has (arithmetic) genus zero. This implies that there are no closed circuits.
- Each component has at least 3 special (singular or marked) points, and no marked point is singular.

- Two stable curves are said to be n-isomorphic if there is an isomorphism preserving the markings.

- Two stable curves are said to be n-isomorphic if there is an isomorphism preserving the markings.
- This isomorphism necessarily preserves the singularities!

- Two stable curves are said to be n-isomorphic if there is an isomorphism preserving the markings.
- This isomorphism necessarily preserves the singularities!
- Stable n-curves have no non-trivial automorphisms preserving the markings (no n-automorphisms).

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

- $\overline{M_{0,n}}$ is smooth, compact, and projective!

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

- $\overline{M_{0,n}}$ is smooth, compact, and projective!
- $M_{0,n}$ is open in $\overline{M_{0,n}}$.

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

- $\overline{M_{0,n}}$ is smooth, compact, and projective!
- $M_{0,n}$ is open in $\overline{M_{0,n}}$.
- $\overline{M_{0,n}} - M_{0,n}$ (closed) is a reducible divisor, and any two of its components intersect transversally.

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

- $\overline{M_{0,n}}$ is smooth, compact, and projective!
- $M_{0,n}$ is open in $\overline{M_{0,n}}$.
- $\overline{M_{0,n}} - M_{0,n}$ (closed) is a reducible divisor, and any two of its components intersect transversally.
- The universal family $\overline{U_n}$ is $\overline{M_{0,n+1}}$! (more details on this later).

The Moduli Space $\overline{M_{0,n}}$

There is a (fine) moduli space for stable n-curves for $n \geq 3$ (!)
Moreover:

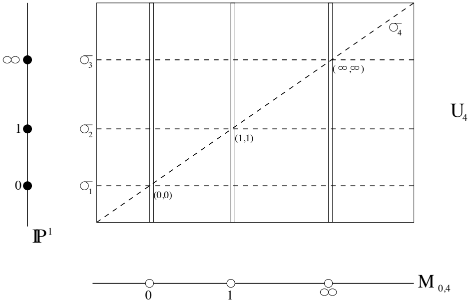
- $\overline{M_{0,n}}$ is smooth, compact, and projective!
- $M_{0,n}$ is open in $\overline{M_{0,n}}$.
- $\overline{M_{0,n}} - M_{0,n}$ (closed) is a reducible divisor, and any two of its components intersect transversally.
- The universal family $\overline{U_n}$ is $\overline{M_{0,n+1}}$! (more details on this later).

Some History

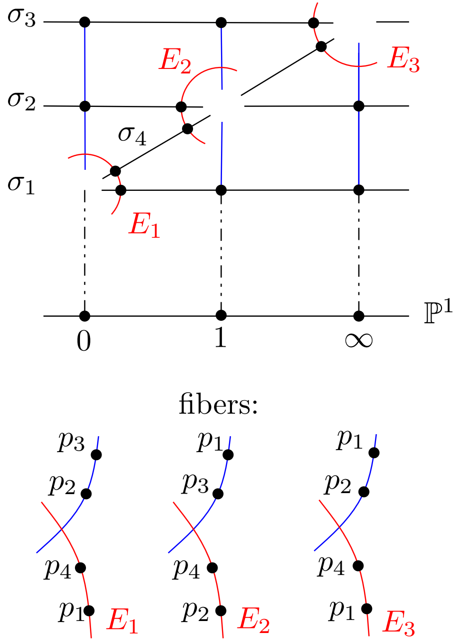
It was first constructed by Knudsen in 1983. One of the highlights of modern algebraic geometry.

$\overline{M_{0,4}}$

$M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ and its universal family
$U_4 \subset M_{0,4} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$



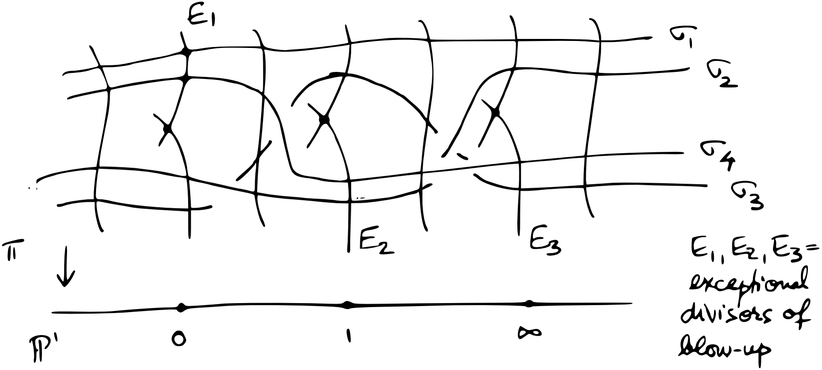
- Blow-up $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $0 \times 0, 1 \times 1, \infty \times \infty$.
- This separates the sections and so we get a family of curves over $\mathbb{P}^1$ where all the fibers now have 4 distinct marked points.



σ3
σ2
σ1
σ4
E2
E3
E1
ℙ1
0
1
∞
fibers:
p3
p2
p4
p1
E1
p1
p3
p4
p2
E2
p1
p2
p4
p1
E3

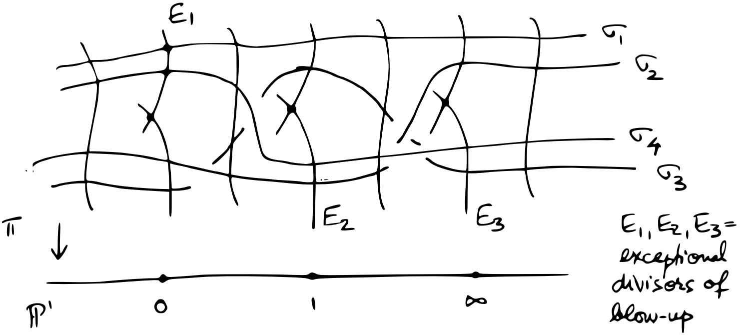
This is $\overline{M_{0,4}} = \mathbb{P}^1$ together with its universal family $\overline{U_4}$.

$$\overline{U_4} = Bl_{3 \text{ points}}\, \mathbb{P}^1 \times \mathbb{P}^1$$



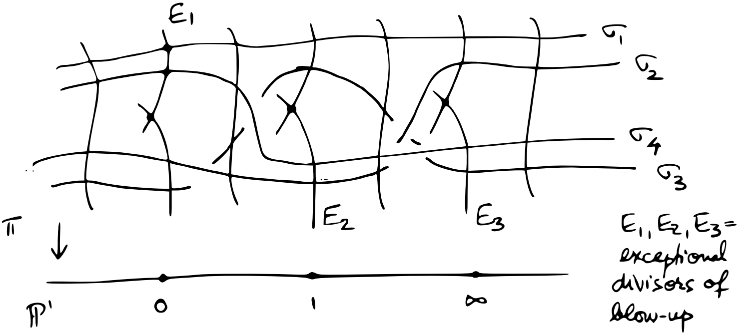
(picture by Ana-Maria)

The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$



Take a point $p \in \overline{U_4}$ (we will find a stable 5-curve is corresponds to).

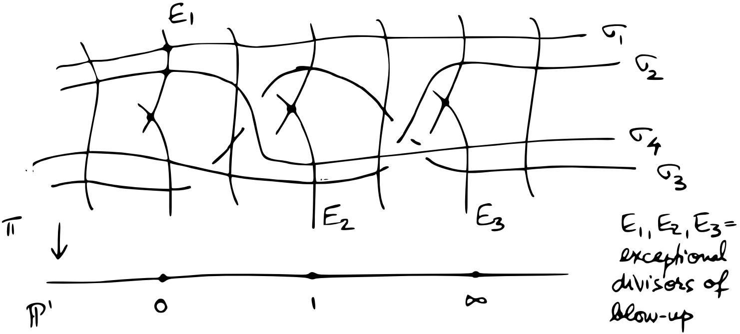
The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$



Take a point $p \in \overline{U_4}$ (we will find a stable 5-curve is corresponds to).

- p is in some fiber of the map, say above q.

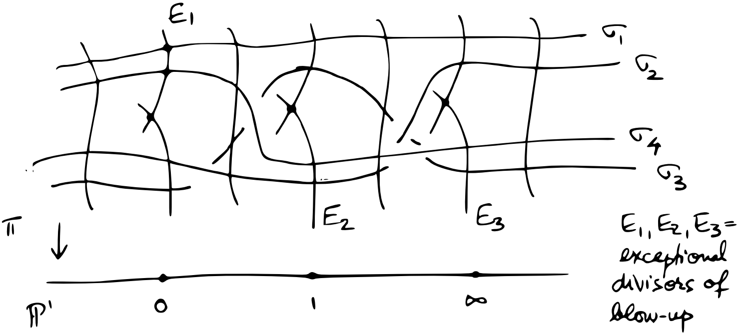
The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$



Take a point $p \in \overline{U_4}$ (we will find a stable 5-curve is corresponds to).

- p is in some fiber of the map, say above q.
- If it is not in one of the sections, and not one of the singular points of the special fibers, then the stable 5-curve it corresponds to is the fiber with the marked points $(\sigma_1(q), \ldots, \sigma_4(q), p = p_5)$.

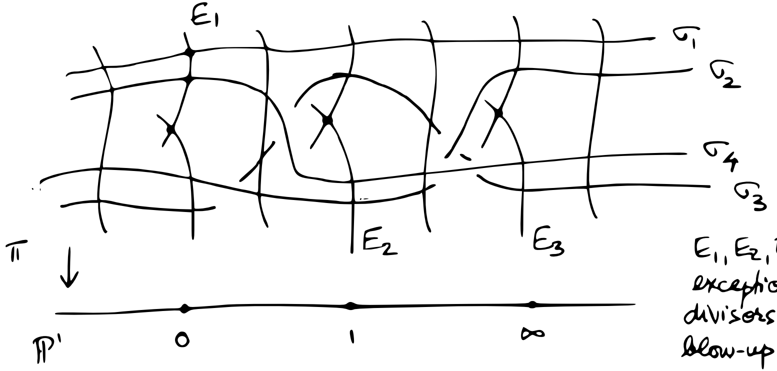
The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$



Take a point $p \in \overline{U_4}$ (we will find a stable 5-curve is corresponds to).

- p is in some fiber of the map, say above q.
- If it is not in one of the sections, and not one of the singular points of the special fibers, then the stable 5-curve it corresponds to is the fiber with the marked points $(\sigma_1(q), \ldots, \sigma_4(q), p = p_5)$.
- Otherwise we need to *stabilize* the curve.

The universal family $\overline{U_4}$ is $\overline{M_{0,5}}$



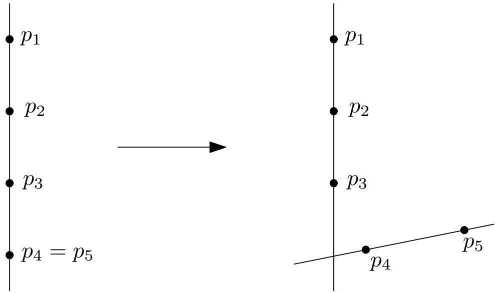
E1, E2, E3 = exceptional divisors of blow-up

Stabilization

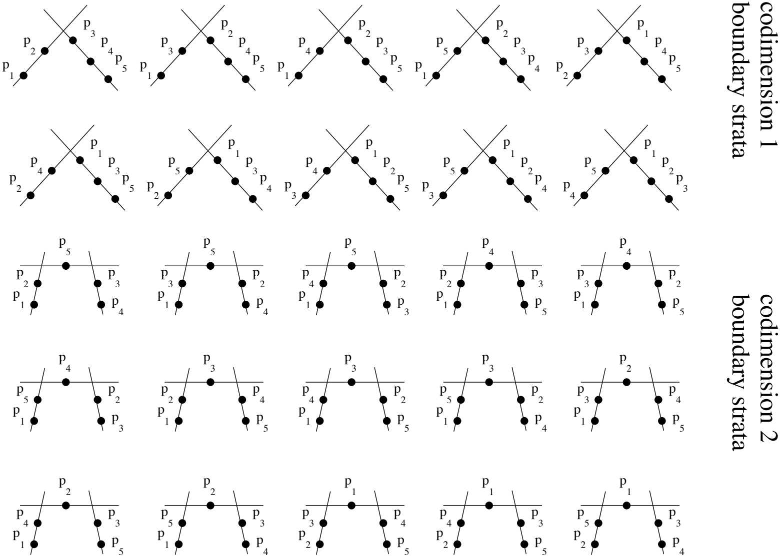
- If p is not in one of the sections, but it is a singular point of one of the special ones, then replace the singular point by a $\mathbb{P}^1$ and place $p = p_5$ in this new $\mathbb{P}^1$.

Stabilization

- If p is in one of the sections, say σ_4 then take the fiber and add a $\mathbb{P}^1$ at $\sigma_4(q)$ and put p_4 and p_5 on this new $\mathbb{P}^1$ (this is unique up to isomorphism).



This takes care of all isomorphism classes of 5-curves!



(picture taken from Renzo's notes)

The Universal Family for general n

For each $i = 1, \ldots, n+1$ there is a morphism

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

that forgets the information of the i-th marking (a forgetful morphism).

The Universal Family for general n

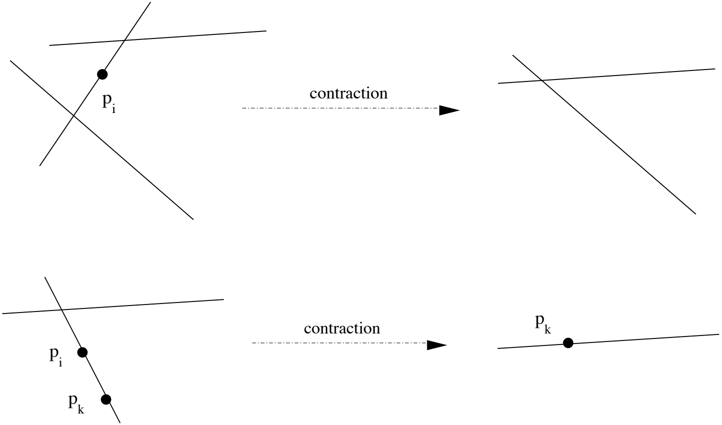
For each $i = 1, \ldots, n+1$ there is a morphism

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

that forgets the information of the i-th marking (a forgetful morphism).

The map stabilizes the curve if it becomes unstable by contracting the component that became unstable.

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$



Why this would be expected to be a morphism of varieties is way beyond my comprehension!

Each of these

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

identifies the universal family $\overline{U_n}$ with $\overline{M_{0,n+1}}$

Each of these

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

identifies the universal family $\overline{U_n}$ with $\overline{M_{0,n+1}}$

This is precisely what we saw for $\overline{U_4} \to \overline{M_{0,4}}$ where we are forgetting p_5 (so the map is π_5)...the only thing is that we are now thinking backwards!

Each of these

$$\pi_i : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

identifies the universal family $\overline{U_n}$ with $\overline{M_{0,n+1}}$

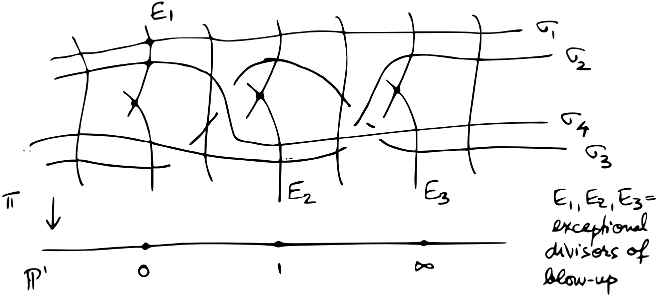
This is precisely what we saw for $\overline{U_4} \to \overline{M_{0,4}}$ where we are forgetting p_5 (so the map is π_5)...the only thing is that we are now thinking backwards!

(Instead of taking a point in $\overline{U_4}$ and understanding what stable 5-curves it parametrizes, we are taking a 5-curve, and seeing what 4-curve we can get from it.)

$\overline{M_{0,4}} = \mathbb{P}^1$ together with its universal family $\overline{U_4} = \overline{M_{0,5}}$.

$$\pi = \pi_5 : \overline{M_{0,5}} \to \overline{M_{0,4}}$$

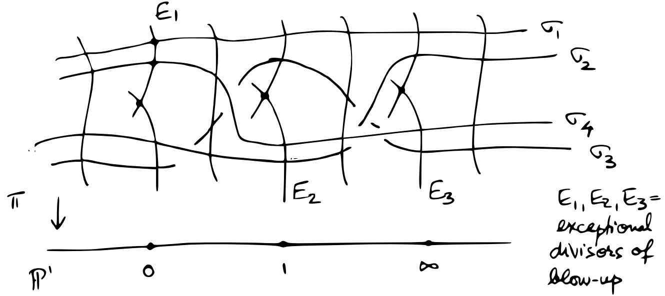
forgets the 5-th point.



$\overline{M_{0,4}} = \mathbb{P}^1$ together with its universal family $\overline{U_4} = \overline{M_{0,5}}$.

$$\pi = \pi_5 : \overline{M_{0,5}} \to \overline{M_{0,4}}$$

forgets the 5-th point.



The contractions of the 5-curves are the fibers of the map in the picture!

The sections of the universal family

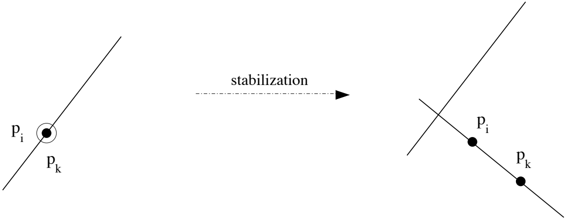
The sections of the universal family represented as

$$\pi_k : \overline{M_{0,n+1}} \to \overline{M_{0,n}}$$

that give the marked points are given by

$$\sigma_i : \overline{M_{0,n}} \to \overline{M_{0,n+1}}$$

by setting $p_i = p_k$ and stabilizing.

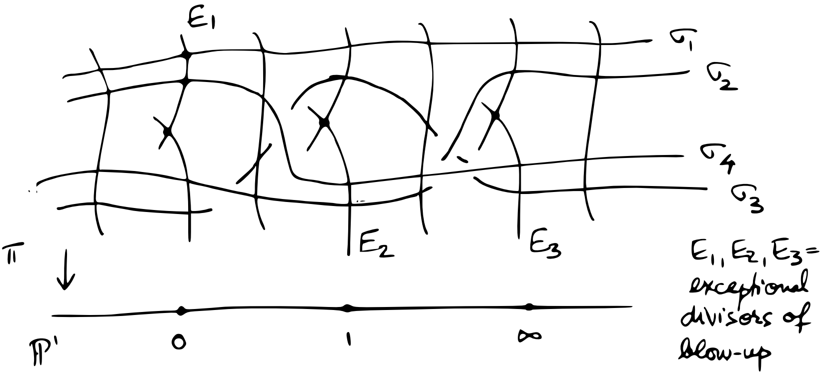


(picture taken from Renzo's notes)

Note that $k = i$ won't happen! We only need n sections.

The sections of the universal family

This may seem strange, but remember that the sections of $\overline{U_4} \to \overline{M_{0,4}}$ were the points of $\overline{U_4}$ that corresponded to (almost all) reducible 5-stable curves!



Knudsen's Construction

- Is inductive.

Knudsen's Construction

- Is inductive.
- Uses the identification $\overline{U_n} = \overline{M_{0,n+1}}$ and constructs the fiber product

$$\begin{array}{ccc} \star & \longrightarrow & \overline{U_n} = \overline{M_{0,n+1}} \\ \downarrow & & \downarrow \\ \overline{M_{0,n+1}} & \longrightarrow & \overline{M_{0,n}} \end{array}$$

Knudsen's Construction

- Is inductive.
- Uses the identification $\overline{U_n} = \overline{M_{0,n+1}}$ and constructs the fiber product

$$\begin{array}{ccc} \star & \longrightarrow & \overline{U_n} = \overline{M_{0,n+1}} \\ \downarrow & & \downarrow \\ \overline{M_{0,n+1}} & \longrightarrow & \overline{M_{0,n}} \end{array}$$

- on the right we have the sections $\sigma_1, \ldots, \sigma_n$ and on the left we have these plus the diagonal Δ which intersects all of them.

Knudsen's Construction

- Is inductive.
- Uses the identification $\overline{U_n} = \overline{M_{0,n+1}}$ and constructs the fiber product

$$\begin{array}{ccc} \star & \longrightarrow & \overline{U_n} = \overline{M_{0,n+1}} \\ \downarrow & & \downarrow \\ \overline{M_{0,n+1}} & \longrightarrow & \overline{M_{0,n}} \end{array}$$

- on the right we have the sections $\sigma_1, \ldots, \sigma_n$ and on the left we have these plus the diagonal Δ which intersects all of them.
- This $\star$ space may be singular! [IS IT?]

Knudsen's Construction

- Is inductive.
- Uses the identification $\overline{U_n} = \overline{M_{0,n+1}}$ and constructs the fiber product

$$\begin{array}{ccc} \star & \longrightarrow & \overline{U_n} = \overline{M_{0,n+1}} \\ \downarrow & & \downarrow \\ \overline{M_{0,n+1}} & \longrightarrow & \overline{M_{0,n}} \end{array}$$

- on the right we have the sections $\sigma_1, \ldots, \sigma_n$ and on the left we have these plus the diagonal Δ which intersects all of them.
- This $\star$ space may be singular! [IS IT?]
- Knudsen proves that

$$\overline{M_{0,n+2}} = \overline{U_{n+1}} = \left\{ \begin{array}{c} \text{minimal desingularization of} \\ \star \text{ that separates the sections} \end{array} \right\}$$

Kapranov's Construction of $\overline{M_{0,n}}$ (1993)

Kapranov interprets the points in $\overline{M_{0,n}}$ as the rational normal curves in $\mathbb{P}^{n-2}$ through n points in general position, together with nodal degenerations of these curves.

Kapranov's Construction of $\overline{M_{0,n}}$ (1993)

Kapranov interprets the points in $\overline{M_{0,n}}$ as the rational normal curves in $\mathbb{P}^{n-2}$ through n points in general position, together with nodal degenerations of these curves.

Rational Normal Curve (Veronese Curve) in $\mathbb{P}^n$

Kapranov's Construction of $\overline{M_{0,n}}$ (1993)

Kapranov interprets the points in $\overline{M_{0,n}}$ as the rational normal curves in $\mathbb{P}^{n-2}$ through n points in general position, together with nodal degenerations of these curves.

Rational Normal Curve (Veronese Curve) in $\mathbb{P}^n$

Is the image of $\mathbb{P}^1$ in $\mathbb{P}^n$ under a morphism

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^n \\ [s:t] & \mapsto & [f_0(s,t) : \ldots : f_n(s,t)] \end{array}$$

where $f_0, \ldots, f_n$ is a basis for the vector space of degree n homogeneous polynomials of degree n.

Kapranov's Construction of $\overline{M_{0,n}}$ (1993)

Kapranov interprets the points in $\overline{M_{0,n}}$ as the rational normal curves in $\mathbb{P}^{n-2}$ through n points in general position, together with nodal degenerations of these curves.

Rational Normal Curve (Veronese Curve) in $\mathbb{P}^n$

Is the image of $\mathbb{P}^1$ in $\mathbb{P}^n$ under a morphism

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^n \\ [s:t] & \mapsto & [f_0(s,t):\ldots:f_n(s,t)] \end{array}$$

where $f_0, \ldots, f_n$ is a basis for the vector space of degree n homogeneous polynomials of degree n. After a change in coordinates in $\mathbb{P}^n$ one can obviously take it the the curve

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^n \\ [s:t] & \mapsto & [s^n : s^{n-1}t : \ldots : t^n] \end{array}$$

i.e., the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$.

- (Classical) There is a unique rational normal curve (Veronese curve) in $\mathbb{P}^{n-2}$ through $n+1$ points in general position.

- (Classical) There is a unique rational normal curve (Veronese curve) in $\mathbb{P}^{n-2}$ through $n+1$ points in general position.
- e.g. A unique conic in $\mathbb{P}^2$ through 5 points general points, a unique twisted cubic in $\mathbb{P}^3$ through 6 general points, etc.

- (Classical) There is a unique rational normal curve (Veronese curve) in $\mathbb{P}^{n-2}$ through $n+1$ points in general position.
- e.g. A unique conic in $\mathbb{P}^2$ through 5 points general points, a unique twisted cubic in $\mathbb{P}^3$ through 6 general points, etc.
- Kapranov: If you drop one of these points, the the family of Veronese curves and their nodal degenerations (more on this later) is a representation of all n-stable curves, where the markings correspond to the points that the curves are going through.

- (Classical) There is a unique rational normal curve (Veronese curve) in $\mathbb{P}^{n-2}$ through $n+1$ points in general position.
- e.g. A unique conic in $\mathbb{P}^2$ through 5 points general points, a unique twisted cubic in $\mathbb{P}^3$ through 6 general points, etc.
- Kapranov: If you drop one of these points, the the family of Veronese curves and their nodal degenerations (more on this later) is a representation of all n-stable curves, where the markings correspond to the points that the curves are going through.

Example $n = 4$

The curves that $\overline{M_{0,4}} = \mathbb{P}^1$ parametrizes can be realized as the set of conics in $\mathbb{P}^2$ through 4 points in general position, together with their nodal degenerations.

- (Classical) There is a unique rational normal curve (Veronese curve) in $\mathbb{P}^{n-2}$ through $n+1$ points in general position.
- e.g. A unique conic in $\mathbb{P}^2$ through 5 points general points, a unique twisted cubic in $\mathbb{P}^3$ through 6 general points, etc.
- Kapranov: If you drop one of these points, the the family of Veronese curves and their nodal degenerations (more on this later) is a representation of all n-stable curves, where the markings correspond to the points that the curves are going through.

Example $n = 4$

The curves that $\overline{M_{0,4}} = \mathbb{P}^1$ parametrizes can be realized as the set of conics in $\mathbb{P}^2$ through 4 points in general position, together with their nodal degenerations.

What is happening here: as the conic changes, the cross ratio of the points is changing (after identifying the conic with $\mathbb{P}^1$).

For example, if we let the points be $(0, 0), (0, 1), (1, 0), (1, 1)$, then the general conic through these four points is of the form

$$ax^2 + by^2 - ax - by = 0$$

for $[a : b] \in \mathbb{P}^1$.

For example, if we let the points be $(0,0),(0,1),(1,0),(1,1)$, then the general conic through these four points is of the form

$$ax^2 + by^2 - ax - by = 0$$

for $[a:b] \in \mathbb{P}^1$.

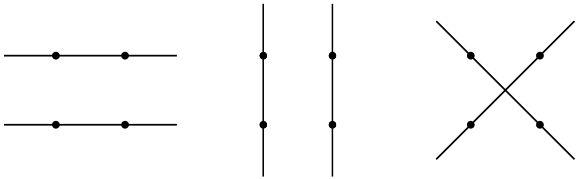
The determinant of the associated quadratic form is $-ab(a+b)/4$, and so one can explicitly see that there are only 3 singular conics through these points, given by the values $[a:b] = [0:1],[1:0],[1:-1]$.

For example, if we let the points be $(0,0),(0,1),(1,0),(1,1)$, then the general conic through these four points is of the form

$$ax^2 + by^2 - ax - by = 0$$

for $[a : b] \in \mathbb{P}^1$.

The determinant of the associated quadratic form is $-ab(a+b)/4$, and so one can explicitly see that there are only 3 singular conics through these points, given by the values
$[a : b] = [0 : 1], [1 : 0], [1 : -1]$.



The total family of this pencil of conics is

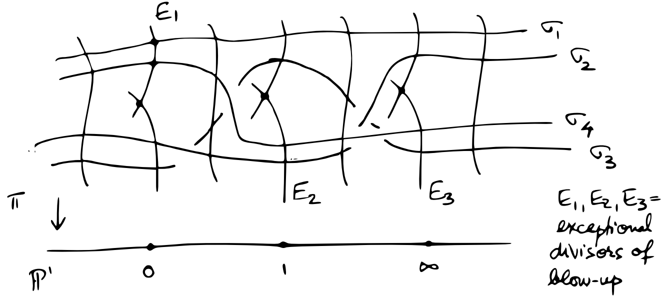
$$\begin{array}{c} U \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{a,b} \\ \big\downarrow \\ \mathbb{P}^1_{a,b} \end{array}$$

where $U = \{[x : y : z] \times [a : b] \mid ax^2 + by^2 - axz - byz = 0\}$.

The total family of this pencil of conics is

$$\begin{array}{c} U \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{a,b} \\ \big\downarrow \\ \mathbb{P}^1_{a,b} \end{array}$$

where $U = \{[x : y : z] \times [a : b] \mid ax^2 + by^2 - axz - byz = 0\}$. This explicitly shows the situation depicted in our picture (one can replace b by $-b$ to have the fibers correspond exactly)



Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Then,

- $V(p_0, \ldots, p_n) \cong M_{0,n}$.

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Then,

- $V(p_0, \ldots, p_n) \cong M_{0,n}$.
- The closure of $V(p_0, \ldots, p_n)$ in $\mathcal{H}$ is isomorphic to $\overline{M_{0,n}}$.

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Then,

- $V(p_0, \ldots, p_n) \cong M_{0,n}$.
- The closure of $V(p_0, \ldots, p_n)$ in $\mathcal{H}$ is isomorphic to $\overline{M_{0,n}}$.
- The points of the closure in $\mathcal{H}$ correspond (as parametrized by the Hilbert Scheme) precisely to stable n-curves in $\mathbb{P}^{n-2}$ going through the p_i. This correspondence agrees with the one from $\overline{M_{0,n}}$.

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Then,

- $V(p_0, \ldots, p_n) \cong M_{0,n}$.
- The closure of $V(p_0, \ldots, p_n)$ in $\mathcal{H}$ is isomorphic to $\overline{M_{0,n}}$.
- The points of the closure in $\mathcal{H}$ correspond (as parametrized by the Hilbert Scheme) precisely to stable n-curves in $\mathbb{P}^{n-2}$ going through the p_i. This correspondence agrees with the one from $\overline{M_{0,n}}$.
- Each component of these reducible curves is a Veronese curve in the linear subspace of $\mathbb{P}^{n-2}$ it spans. If the component contains m special points (marked or singular), then the component is a Veronese curve of degree $m-2$ in a $\mathbb{P}^{m-2}$.

Explicit statement in Kapranov's paper

- Let $p_1, \ldots, p_n$ be n points in general position in $\mathbb{P}^{n-2}$,
- $\mathcal{H}$ be Hilbert Scheme parametrizing all subschemes of $\mathbb{P}^{n-2}$,
- $V(p_0, \ldots, p_n)$ be the locus in $\mathcal{H}$ of all the rational normal curves going through the p_i.

Then,

- $V(p_0, \ldots, p_n) \cong M_{0,n}$.
- The closure of $V(p_0, \ldots, p_n)$ in $\mathcal{H}$ is isomorphic to $\overline{M_{0,n}}$.
- The points of the closure in $\mathcal{H}$ correspond (as parametrized by the Hilbert Scheme) precisely to stable n-curves in $\mathbb{P}^{n-2}$ going through the p_i. This correspondence agrees with the one from $\overline{M_{0,n}}$.
- Each component of these reducible curves is a Veronese curve in the linear subspace of $\mathbb{P}^{n-2}$ it spans. If the component contains m special points (marked or singular), then the component is a Veronese curve of degree $m-2$ in a $\mathbb{P}^{m-2}$.
- The analogous statement holds if instead of $\mathcal{H}$ one uses the Chow variety.

some comments...

Kapranov (below the statement):

"The first statement $[V(p_0, \ldots, p_n) \cong M_{0,n}]$ is classical. This makes it rather surprising that the space $\overline{M_{0,n}}$ was not discovered by classical algebraic geometers."

some comments...

Kapranov (below the statement):

"The first statement $[V(p_0, \ldots, p_n) \cong M_{0,n}]$ is classical. This makes it rather surprising that the space $\overline{M_{0,n}}$ was not discovered by classical algebraic geometers."

Key Point of First Statement:

If $C, C' \in V(p_0, \ldots, p_n)$ are isomorphic as n-curves, the isomorphism $C \to C'$, which necessarily fixes the p_i extends to an isomorphism $\mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$ (this uses duality!) and so is the identity since it fixes n points in general position.

some comments...

Where the embedding comes from:

- A stable n-curve $(C, x_1, \ldots, x_n)$ comes with an invertible sheaf Ω_C.

some comments...

Where the embedding comes from:

- A stable n-curve $(C, x_1, \ldots, x_n)$ comes with an invertible sheaf Ω_C.
- $\Omega_C =$ differentials that are regular in the smooth locus of C, and have at most simple poles at the singular points, with the residues agreeing on the different branches at the singular curve.

some comments...

Where the embedding comes from:

- A stable n-curve $(C, x_1, \ldots, x_n)$ comes with an invertible sheaf Ω_C.
- $\Omega_C =$ differentials that are regular in the smooth locus of C, and have at most simple poles at the singular points, with the residues agreeing on the different branches at the singular curve.
- $\Omega_C(x_1 + \ldots + x_n)$ is very ample and embeds C in $\mathbb{P}^{n-2}$ (this was proved by Knudsen) as a union of Veronese curves where the points x_i get sent to points in general position (proved by Kapranov).

The universal family morphism

Kapranov's indentification of stable n-curves with Veronese curves allowed him to see the forgetful morphism in a new light:

$$\begin{array}{c} \overline{M_{0,n+1}} \\ \downarrow \pi_{n+1} \\ \overline{M_{0,n}} \end{array}$$

The universal family morphism

Kapranov's indentification of stable n-curves with Veronese curves allowed him to see the forgetful morphism in a new light:

$$\begin{array}{c} \overline{M_{0,n+1}} \\ \Big\downarrow \pi_{n+1} \\ \overline{M_{0,n}} \end{array}$$

Fix the $p_1, \ldots, p_{n+1} \in \mathbb{P}^{n-1}$ in general position, and for $p \in \overline{M_{0,n+1}}$ let $C_p \subset \mathbb{P}^{n-1}$ be the curve through $p_1, \ldots, p_{n+1}$ it corresponds to.

The universal family morphism

Kapranov's indentification of stable n-curves with Veronese curves allowed him to see the forgetful morphism in a new light:

$$\begin{array}{c} \overline{M_{0,n+1}} \\ \downarrow {\scriptstyle \pi_{n+1}} \\ \overline{M_{0,n}} \end{array}$$

Fix the $p_1, \ldots, p_{n+1} \in \mathbb{P}^{n-1}$ in general position, and for $p \in \overline{M_{0,n+1}}$ let $C_p \subset \mathbb{P}^{n-1}$ be the curve through $p_1, \ldots, p_{n+1}$ it corresponds to.

Projection from $\mathbb{P}^{n-1}$ form p_{n+1} onto $\mathbb{P}^{n-2}$ maps $p_1, \ldots, p_n$ to points in general position, and it maps $C_p - \{p_{n+1}\}$ to a stable curve whose closure is a point in $\overline{M_{0,n}}$.

The universal family morphism

Kapranov's indentification of stable n-curves with Veronese curves allowed him to see the forgetful morphism in a new light:

$$\begin{array}{c} \overline{M_{0,n+1}} \\ \Big\downarrow \pi_{n+1} \\ \overline{M_{0,n}} \end{array}$$

Fix the $p_1, \ldots, p_{n+1} \in \mathbb{P}^{n-1}$ in general position, and for $p \in \overline{M_{0,n+1}}$ let $C_p \subset \mathbb{P}^{n-1}$ be the curve through $p_1, \ldots, p_{n+1}$ it corresponds to.

Projection from $\mathbb{P}^{n-1}$ form p_{n+1} onto $\mathbb{P}^{n-2}$ maps $p_1, \ldots, p_n$ to points in general position, and it maps $C_p - \{p_{n+1}\}$ to a stable curve whose closure is a point in $\overline{M_{0,n}}$.

This map is π_{n+1}!

The universal family morphism

Kapranov's indentification of stable n-curves with Veronese curves allowed him to see the forgetful morphism in a new light:

$$\begin{array}{c} \overline{M_{0,n+1}} \\ \big\downarrow \pi_{n+1} \\ \overline{M_{0,n}} \end{array}$$

Fix the $p_1, \ldots, p_{n+1} \in \mathbb{P}^{n-1}$ in general position, and for $p \in \overline{M_{0,n+1}}$ let $C_p \subset \mathbb{P}^{n-1}$ be the curve through $p_1, \ldots, p_{n+1}$ it corresponds to.

Projection from $\mathbb{P}^{n-1}$ form p_{n+1} onto $\mathbb{P}^{n-2}$ maps $p_1, \ldots, p_n$ to points in general position, and it maps $C_p - \{p_{n+1}\}$ to a stable curve whose closure is a point in $\overline{M_{0,n}}$.

This map is π_{n+1}!

Example: If C_p has only two marked points on the component containing p_{n+1}, then this component is necessarily a line in $\mathbb{P}^{n-1}$ and it gets contracted by the projection!

The Kapranov Map

Fix the $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ from now on, and for $p \in \overline{M_{0,n}}$, let $C_p \subset \mathbb{P}^{n-2}$ be the genus zero stable n-curve through the p_i that it corresponds to.

The Kapranov Map

Fix the $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ from now on, and for $p \in \overline{M_{0,n}}$, let $C_p \subset \mathbb{P}^{n-2}$ be the genus zero stable n-curve through the p_i that it corresponds to.

Kapranov maps: The maps

$$\begin{array}{rcl} \kappa_n : \overline{M_{0,n}} & \to & \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p & \mapsto & T_{p_n} C_p \end{array}$$

where $T_{p_n} C_p$ is the tangent line to C_p at p_n (the p_i will always be smooth points of the C_p).

The Kapranov Map

Fix the $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ from now on, and for $p \in \overline{M_{0,n}}$, let $C_p \subset \mathbb{P}^{n-2}$ be the genus zero stable n-curve through the p_i that it corresponds to.

Kapranov maps: The maps

$$\begin{array}{rcl} \kappa_n : \overline{M_{0,n}} & \to & \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p & \mapsto & T_{p_n} C_p \end{array}$$

where $T_{p_n} C_p$ is the tangent line to C_p at p_n (the p_i will always be smooth points of the C_p).

(There are similar maps for the other p_i, but I don't want to start adding indices in other places.)

$$\begin{aligned} \kappa_n : \overline{M_{0,n}} &\to \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p &\mapsto T_{p_n} C_p \end{aligned}$$

- They are morphisms!

$$\begin{aligned} \kappa_n : \overline{M_{0,n}} &\rightarrow \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p &\mapsto T_{p_n} C_p \end{aligned}$$

- They are morphisms!
- They are blow-up maps!

$$\begin{array}{rcl} \kappa_n : \overline{M_{0,n}} & \to & \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p & \mapsto & T_{p_n} C_p \end{array}$$

- They are morphisms!
- They are blow-up maps!
- The "transition function" $\mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ from the map for p_i and p_j is a standard Cremona transformation.

$$\begin{array}{rcl} \kappa_n : \overline{M_{0,n}} & \to & \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p & \mapsto & T_{p_n} C_p \end{array}$$

- They are morphisms!
- They are blow-up maps!
- The "transition function" $\mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ from the map for p_i and p_j is a standard Cremona transformation.
- Viewing $\overline{M_{0,n}}$ as the universal family over $\overline{M_{0,n-1}}$ by dropping the n-th marking, the $n-1$ sections of the family get contracted by κ_n to $n-1$ points in general position.

$$\begin{aligned} \kappa_n : \overline{M_{0,n}} &\to \mathbb{P}^{n-3} = \left\{ \begin{array}{c} \text{lines through} \\ p_n \text{ in } \mathbb{P}^{n-2} \end{array} \right\} \\ p &\mapsto T_{p_n} C_p \end{aligned}$$

- They are morphisms!
- They are blow-up maps!
- The "transition function" $\mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ from the map for p_i and p_j is a standard Cremona transformation.
- Viewing $\overline{M_{0,n}}$ as the universal family over $\overline{M_{0,n-1}}$ by dropping the n-th marking, the $n-1$ sections of the family get contracted by κ_n to $n-1$ points in general position.
- κ_n is the blow-up of
  - these $n-1$ points
  - then the strict transform of the lines connecting them
  - then the strict transform of the planes containing three of them
  - etc.

So....

Theorem (Kapranov)

$\overline{M_{0,n}}$ is the blow-up of $\mathbb{P}^{n-3}$ along $n-1$ points in general position, then along the strict transforms of the lines joining the points, then along the planes containing three of them, etc.

So....

Theorem (Kapranov)

$\overline{M_{0,n}}$ is the blow-up of $\mathbb{P}^{n-3}$ along $n-1$ points in general position, then along the strict transforms of the lines joining the points, then along the planes containing three of them, etc.

Note: This automatically implies that $\overline{M_{0,n}}$ is smooth, compact and projective.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

$$\begin{array}{ccc} \overline{M_{0,5}} & \xrightarrow{\kappa_5} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi_5} & & \\ \overline{M_{0,4}} = \mathbb{P}^1 & & \end{array}$$

We already knew that $\overline{M_{0,5}}$ was the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points (this is how we originally constructed it).

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

$$\begin{array}{ccc} \overline{M_{0,5}} & \xrightarrow{\kappa_5} & \mathbb{P}^2 \\ \downarrow \pi_5 & & \\ \overline{M_{0,4}} = \mathbb{P}^1 & & \end{array}$$

We already knew that $\overline{M_{0,5}}$ was the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points (this is how we originally constructed it).

Kapranov's result states this by saying that $\overline{M_{0,5}}$ is the blow-up of $\mathbb{P}^2$ at four points in general position (remember that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point is the same as the blow-up of $\mathbb{P}^2$ at two points)

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

Remember:

- Each point in $\overline{M_{0,5}}$ represents a (union of) Veronese curve(s) in $\mathbb{P}^3$ through 5 points $p_1, \ldots, p_5$ in general position.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

Remember:

- Each point in $\overline{M_{0,5}}$ represents a (union of) Veronese curve(s) in $\mathbb{P}^3$ through 5 points $p_1, \dots, p_5$ in general position.
- The Kapranov map
$$\overline{M_{0,5}} \to \mathbb{P}^2$$
sends the point in $\overline{M_{0,5}}$ to the tangent line at p_5 of the curve it corresponds to.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

Remember:

- Each point in $\overline{M_{0,5}}$ represents a (union of) Veronese curve(s) in $\mathbb{P}^3$ through 5 points $p_1, \ldots, p_5$ in general position.
- The Kapranov map
$$\overline{M_{0,5}} \to \mathbb{P}^2$$
sends the point in $\overline{M_{0,5}}$ to the tangent line at p_5 of the curve it corresponds to.
- The Kapranov map is also the blow-up of $\mathbb{P}^2$ at 4 points.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

Remember:

- Each point in $\overline{M_{0,5}}$ represents a (union of) Veronese curve(s) in $\mathbb{P}^3$ through 5 points $p_1, \ldots, p_5$ in general position.
- The Kapranov map
$$\overline{M_{0,5}} \to \mathbb{P}^2$$
sends the point in $\overline{M_{0,5}}$ to the tangent line at p_5 of the curve it corresponds to.
- The Kapranov map is also the blow-up of $\mathbb{P}^2$ at 4 points.

Question:

What are the curves in the exceptional divisors?

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

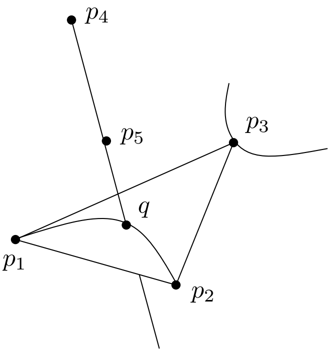
A $\mathbb{P}^1$ collapsed by κ_5:

- Let p_4p_5 be the line joining p_4 and p_3 in $\mathbb{P}^3$.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

A $\mathbb{P}^1$ collapsed by κ_5:

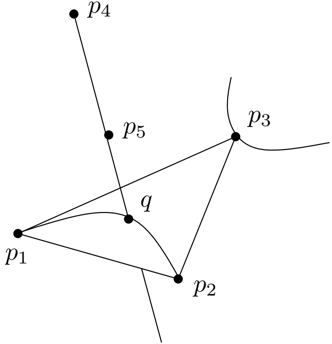
- Let p_4p_5 be the line joining p_4 and p_3 in $\mathbb{P}^3$.
- This intersects the plane $p_1p_2p_3$ spanned by p_1, p_2, p_3 at a point q, and p_1, p_2, p_3, q are 4 points in general position in $\mathbb{P}^2 = p_1p_2p_3$.



Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

A $\mathbb{P}^1$ collapsed by κ_5:

- Let p_4p_5 be the line joining p_4 and p_3 in $\mathbb{P}^3$.
- This intersects the plane $p_1p_2p_3$ spanned by p_1, p_2, p_3 at a point q, and p_1, p_2, p_3, q are 4 points in general position in $\mathbb{P}^2 = p_1p_2p_3$.

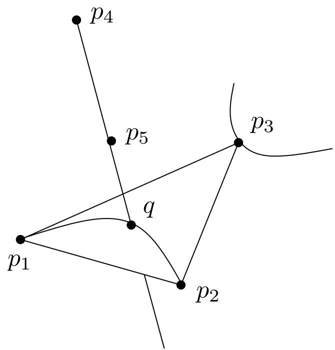


- There is a family of conics in $\mathbb{P}^2 = p_1p_2p_3$ through the 4 points p_1, p_2, p_3, q.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

A $\mathbb{P}^1$ collapsed by κ_5:

- Let p_4p_5 be the line joining p_4 and p_3 in $\mathbb{P}^3$.
- This intersects the plane $p_1p_2p_3$ spanned by p_1, p_2, p_3 at a point q, and p_1, p_2, p_3, q are 4 points in general position in $\mathbb{P}^2 = p_1p_2p_3$.

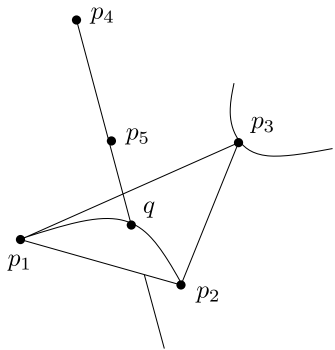


- There is a family of conics in $\mathbb{P}^2 = p_1p_2p_3$ through the 4 points p_1, p_2, p_3, q.
- This family is parametrized by $\mathbb{P}^1$, so correponds to a $\mathbb{P}^1 \subset \overline{M_{0,5}}$.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

A $\mathbb{P}^1$ collapsed by κ_5:

- Let p_4p_5 be the line joining p_4 and p_3 in $\mathbb{P}^3$.
- This intersects the plane $p_1p_2p_3$ spanned by p_1, p_2, p_3 at a point q, and p_1, p_2, p_3, q are 4 points in general position in $\mathbb{P}^2 = p_1p_2p_3$.



- There is a family of conics in $\mathbb{P}^2 = p_1p_2p_3$ through the 4 points p_1, p_2, p_3, q.
- This family is parametrized by $\mathbb{P}^1$, so correponds to a $\mathbb{P}^1 \subset \overline{M_{0,5}}$.
- All the curves in this family have the same tangent line at p_5, and so get sent to the same point by κ_5!

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

The other exceptional $\mathbb{P}^1$'s of

$$\overline{M_{0,5}} \to \mathbb{P}^2$$

are constructed similarly, starting with the lies p_1p_5, p_2p_5, p_3p_5 respectively.

Example with $\overline{M_{0,4}}$ and $\overline{M_{0,5}}$

The other exceptional $\mathbb{P}^1$'s of

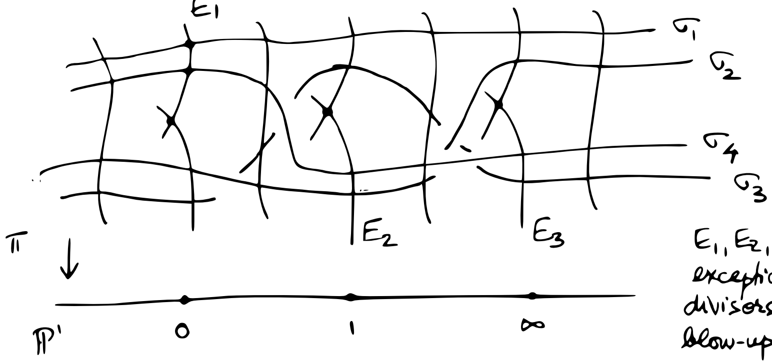
$$\overline{M_{0,5}} \to \mathbb{P}^2$$

are constructed similarly, starting with the lies p_1p_5, p_2p_5, p_3p_5 respectively.

We explicitly see they are the sections $\sigma_1, \ldots, \sigma_4$ of the universal family.

$$\begin{array}{ccc} \overline{M_{0,5}} & \xrightarrow{\kappa_5} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi_5} & & \\ \overline{M_{0,4}} = \mathbb{P}^1 & & \end{array}$$

Thus, κ_5 is sort of the horizontal projection in Ana-Maria's picture



E_1, E_2, E_3 = exceptional divisors of blow-up

Compare also to the explicit pencil of conics through 4 points we found (all maps are projections)

$$\begin{array}{ccc} U \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{a,b} & \longrightarrow & \mathbb{P}^2_{x,y,z} \\ \downarrow & & \\ \mathbb{P}^1_{a,b} & & \end{array}$$

where $U = \{[x : y : z] \times [a : b] \mid ax^2 + by^2 - axz - byz = 0\}$.

Compare also to the explicit pencil of conics through 4 points we found (all maps are projections)

$$\begin{array}{ccc} U \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{a,b} & \longrightarrow & \mathbb{P}^2_{x,y,z} \\ \downarrow & & \\ \mathbb{P}^1_{a,b} & & \end{array}$$

where $U = \{[x : y : z] \times [a : b] \mid ax^2 + by^2 - axz - byz = 0\}$.

The horizontal (projection) map is κ_5!!

How the Kapranov maps allow one to see the stable curves we are parametrizing

If $p \in \overline{M_{0,4}}$, then it corresponds to a fiber of the vertical map

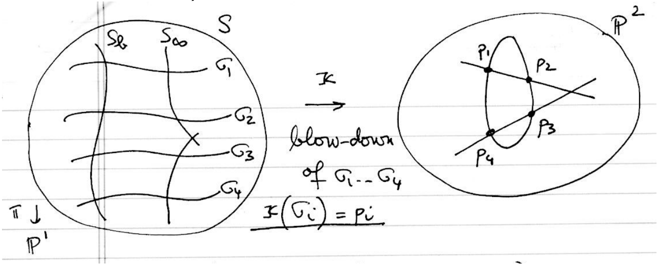
$$\begin{array}{ccc} \overline{M_{0,5}} & \xrightarrow{\kappa_5} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi_5} & & \\ \overline{M_{0,4}} = \mathbb{P}^1 & & \end{array}$$

How the Kapranov maps allow one to see the stable curves we are parametrizing

If $p \in \overline{M_{0,4}}$, then it corresponds to a fiber of the vertical map

$$\begin{array}{ccc} \overline{M_{0,5}} & \xrightarrow{\kappa_5} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi_5} & & \\ \overline{M_{0,4}} = \mathbb{P}^1 & & \end{array}$$

The horizontal map takes this fiber to an actual conic in $\mathbb{P}^2$ through 4 points, which is the Veronese curve corresponds to! (picture by Ana-Maria)

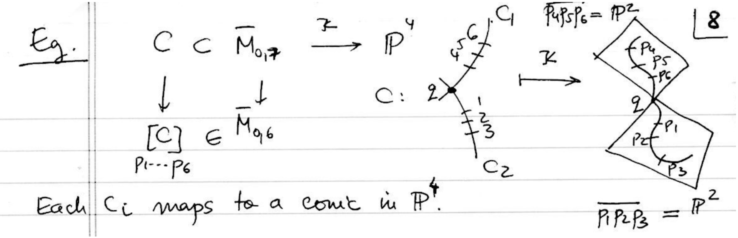


This works in general!

The image under the horizontal map of a fiber above p of the vertical map in

$$\begin{array}{ccc} \overline{M_{0,n+1}} & \xrightarrow{\kappa_{n+1}} & \mathbb{P}^{n-2} \\ \pi_{n+1}\big\downarrow & & \\ p \in \overline{M_{0,n}} & & \end{array}$$

is a model for the of the stable n-curve that p corresponds to.



(picture by Ana-Maria). Note, the C_i are conics because the contain 4 special points each.

The End

References

- Mikhail Kapranov:
  - "Veronese curves and Grothendieck-Knudsen moduli space M(0,n)", Journal of Algebraic Geometry, 1993.
  - "Chow quotient of Grassmannians I", Adv. Soviet Math.
- Ana-Maria Castravet: Course, and course notes available online, "Topics in Geometry and Topology (Moduli of curves)", Spring 2010.
- Renzo Cavalieri: Course notes available online, "Math 676 Topics: Moduli Spaces", Fall 2010.