Ordinals and Cardinals

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Cantor's Ordinals: How he found them

Definition

Let $A \subseteq \mathbb{R}$. We say $x \in A$ is a limit point if it is not isolated.

$$A' = \text{The set of limit points of A}$$

Then

$$A \supseteq A' \supseteq A'' \supseteq A^{(3)} \supseteq A^{(4)} \supseteq \dots$$

$$A^{(n)} := \text{The ``}n\text{-th order'' limit points}$$

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Example

$$\begin{aligned} A &= \{0\} \cup \{1/n \mid n \in \mathbb{N}^*\} \\ A^{(1)} &= \{0\} \\ A^{(2)} &= \emptyset \\ &\vdots \end{aligned}$$

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$$\begin{aligned} B &= A \cup [1,2] \\ B^{(1)} &= \{0\} \cup [1,2] \\ B^{(2)} &= [1,2] \\ B^{(3)} &= [1,2] \\ &\vdots \end{aligned}$$

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Exercise

Show that for any n there is an $A \subseteq \mathbb{R}$ for which

$$A \supset A^{(1)} \supset A^{(2)} \supset A^{(3)} \supset^{(4)} \supset \ldots \supset A^{(n)}$$

is strict.

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Question

¿ Are there sets for which the sequence

$$A \supset A' \supset A'' \supset A^{(3)} \supset A^{(4)} \supset \dots$$

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$$A^{(\infty)} = \bigcap A^{(n)}.$$

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Question

¿What about the limit points of $A^{(\infty)}$?

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Cantor:

$$\begin{aligned} A^{(\infty+1)} &= \left(A^{(\infty)}\right)' \\ A^{(\infty+2)} &= \left(A^{(\infty)}\right)'' \\ A^{(\infty+3)} &= \left(A^{(\infty)}\right)^{(3)} \\ &\vdots \end{aligned}$$

Modern Notation

$$\begin{aligned} \omega &:= (\infty) \\ \omega+1 &:= (\infty+1) \\ \omega+2 &:= (\infty+2) \\ &\vdots \end{aligned}$$

The Sequence of Ordinals

$1, 2, 3, 4, 5, \ldots$

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$1, 2, 3, 4, 5, \ldots, \omega$

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$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots,\ \omega\ ,\ \omega+1,\ \omega+2,\ \omega+3\ ,\ \ldots$$

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$$1, 2, 3, 4, 5, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots, \omega+\omega$$

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$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ ,\ \omega\ ,\ \omega+1,\ \omega+2,\ \omega+3\ ,\ \ldots\ ,\ \omega+\omega =: \omega \cdot 2$$

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$$\omega\cdot 4$$

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$$\omega^2+\omega+1,\ \ldots,\ \omega^2+\omega\cdot 2$$

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$\ldots,\ \omega^5,\ \ldots,\ \omega^\omega\ ,\ \omega^\omega+1,\ \ldots,\ \omega^{\omega\cdot 2}\ ,\ \omega^{\omega\cdot 2}+1,\ \ldots,\ \omega^{\omega\cdot 2}\ ,\ \ldots,\ ,$
$\omega^{\omega\cdot 4},\ \ldots,\ \omega^{\omega\cdot 5}\ \ldots,\ \omega^{\omega^2}\ ,\ \omega^{\omega^2}+1,\ \ldots,\ \omega^{\omega^3}\ ,\ \ldots,\ \omega^{\omega^4},\ \ldots,\ \omega^{\omega^\omega}\ ,$

$\omega^{\omega^\omega}+1,\ \ldots$

The Sequence of Ordinals

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots,\ \omega\ ,\ \omega+1,\ \omega+2,\ \omega+3\ ,\ \ldots,\ \omega+\omega =: \omega\cdot 2\ ,$$
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\end{gathered}
$$

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…and so on.

What this (nonsense) led to

From here Cantor went on and created:

- Set Theory
- Theory of cardinals and cardinalities
- Theory of well orders.

Using the theory of ordinals and taking succesive limit points he proved:

Theorem

Every subset of $\mathbb{R}$ is either countable or has the cardinality of $\mathbb{R}$.

Modern Definition of Ordinals

Cantor's naive definition of ordinal numbers

- Start with the natural numbers.
- For each ordinal there is a succesor ordinal.
- Least upper bounds exist: For each set of ordinals $\{\alpha_i\}$ there is a least ordinal which is larger than them all ($\sup\{\alpha_i\}$).

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Modern Definition

A set α is an ordinal if

- α is well ordered by $\in$.
- α is transitive.

Modern Definition of Ordinals

Definition

$(A, \leq)$ is a well order if $\leq$ is a linear order on A and every non-empty subset of A has a minimal element.

Example:

$(\mathbb{N}, \leq)$ is a well order (equivalent to the induction principle) while $(\mathbb{Z}, \leq)$ isn't.

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$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is transitive.

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Definition

A set A is well ordered by $\in$ if $(A, \in)$ is well ordered.

The Modern Construction

$$
\begin{aligned}
0 &:= \emptyset \\
1 &:= \{\emptyset\} = \{0\} \\
2 &:= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &:= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
4 &:= \{0, 1, 2, 3\} \\
&\vdots \\
n+1 &:= \{0, 1, 2, \ldots, n\} = n \cup \{n\} \\
&\vdots
\end{aligned}
$$

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\omega &:= \bigcup n \\
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\omega + 2 &:= \omega + 1 \cup \{\omega + 1\} \\
&\vdots \\
\omega + \omega &:= \bigcup \text{all the previous ones.}
\end{aligned}
$$

Note: $1 \in 2 \in 3 \in \ldots \in \omega \in \omega + 1 \in \ldots \in \omega + \omega \in \ldots$

- Each one is transitive. (element implies subset)
- The restriction of $\in$ to any of them gives a well order.

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Theorems from Set Theory

- $\in$ is a linear order on the ordinals (for any two ordinals α and β either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$).
- successor Ordinals: If α is an ordinal, then $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal, and there are no ordinals between α and $\alpha + 1$.
- For each ordinal α, $\alpha = \{\text{ordinals } \beta \mid \beta \in \alpha\}$.
- Supremum of a set of ordinals: If $\{\alpha_i\}$ is a set or ordinals then $\cup\alpha_i$ is an ordinal and is the supremum of the α_i

Cantor's justification for their definition

Definition

Two orders $(A, \leq_A)$, $(B, \leq_B)$ are said to be order isomorphic if there is a bijection $f : A \to B$ with

$$a_1 \leq_A a_2 \text{ if and only if } f(a_1) \leq_B f(a_2)$$

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Theorem (Cantor)

Every well ordered set is order isomorphic to a UNIQUE ordinal.

Consequence

Ordinals can be taken to be canonical representatives of isomorphism classes of well orders.

Examples

$$\begin{array}{rcl} \bullet\bullet\bullet\bullet\ldots & \cong & \omega \\ \bullet\bullet\bullet\bullet\ldots\bullet & \cong & \omega+1 \\ \bullet\bullet\bullet\bullet\ldots\bullet\bullet & \cong & \omega+2 \\ & \vdots & \\ \bullet\bullet\bullet\bullet\ldots\bullet\bullet\bullet\ldots & \cong & \omega+\omega \\ & \vdots & \end{array}$$

Explicitly:

$1 < 3 < 5 < \ldots < 2 < 4 < 6 < \ldots \cong \omega + \omega$

Sharkovskii's Theorem

$$3 \lhd 5 \lhd 7 \ldots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \ldots \lhd 2^2 \cdot 3 \lhd \ldots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

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Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a point of period n ($f^n(x) = x$ and not before) then f has a point of order m for every $m \rhd n$.

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$$(\mathbb{N} - \{\text{powers of two}\}, \lhd) \cong$$

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$$(\mathbb{N} - \{\text{powers of two}\}, \lhd) \cong \omega^2$$

From Ordinals to Cardinals

$$1 < 2 < \ldots \omega < \omega{+}1 \ldots < \omega^\omega < \ldots < \varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} < \varepsilon_0{+}1 < \ldots$$

From Ordinals to Cardinals

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Exercise

They are all countable!

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Exercise

They are all countable!

The first uncountable ordinal

$$w_1 = \bigcup\{\alpha \mid \alpha \text{ is a countable ordinal}\}$$

Then:

- w_1 is an ordinal.
- w_1 is the first uncountable ordinal (almost by definition).

The Alephs

Definition

A Cardinal is an ordinal which is not in bijection with any of its predecessors.

Definition

$$\begin{aligned} \aleph_0 &:= \omega \\ \aleph_1 &:= \omega_1 \\ \aleph_2 &:= \text{The least ordinal which is not in bijection with } \aleph_1 \\ &\vdots \end{aligned}$$

In general for any ordinal α we define

- $\aleph_{\alpha+1} :=$ The least ordinal which is not in bijection with $\aleph_\alpha$.
- $\aleph_\alpha = \bigcup_{\delta<\alpha} \aleph_\delta$ if α is a limit ordinal (not a successor).

The Alephs

Theorem

Any cardinal is one of the alephs.

$\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots < \aleph_\omega := \bigcup_{i<\omega} \aleph_i$

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The Alephs

Theorem

Any cardinal is one of the alephs.

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$$\ldots < \aleph_{\aleph_{\aleph_{\ddots}}} \ (\aleph_0 \text{ times}) < \ldots < \aleph_{\aleph_{\aleph_{\ddots}}} \ (\aleph_1 \text{ times}) < \ldots$$

…and so on.

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...and so on.

NOTE: The last two are solutions to $\alpha = \aleph_\alpha$.

The Continuum Hypothesis

Theorem

(Axiom of Choice implies) Every set can be well ordered, so every set has the cardinality of a unique cardinal number.

Example

There is a well order on the real numbers, so $|\mathbb{R}| = \aleph_\alpha$ for some ordinal α. However, it is impossible to construct or define this order!

Continuum Hypothesis (Cantor)

$$|\mathbb{R}| = \aleph_1$$

The Continuum Hypothesis

Theorem

(Axiom of Choice implies) Every set can be well ordered, so every set has the cardinality of a unique cardinal number.

Example

There is a well order on the real numbers, so $| \mathbb{R} |= \aleph_\alpha$ for some ordinal α. However, it is impossible to construct or define this order!

Continuum Hypothesis (Cantor)

$$| \mathbb{R} |= \aleph_1$$

or

$$2^{\aleph_0} = \aleph_1$$

The Continuum Hypothesis

Definition: $2^{\aleph_\alpha} =$ The cardinal of $\wp(\aleph_\alpha)$

The Generalized Continuum Hypothesis

For any ordinal α:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

Theorem

The continuum hypothesis is independent of ZFC:

$$ZFC + CH \text{ and } ZFC + \neg CH$$

are both consistent assuming ZFC is consistent.

- This is just the second step in the $2^{\aleph_0}$ maths coming from ZF!

Gödel's first incompleteness theorem

Theorem

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

Gödel's first incompleteness theorem

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Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

Consequences

- There are $2^{\aleph_0}$ different maths starting from ZF.

Gödel's first incompleteness theorem

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Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

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- There are $2^{\aleph_0}$ different maths starting from ZF.
- This won't get any better if we change ZF for something else.
- We will never be able to construct a recursive foundation for math with first order logic where every question has an answer.

Gödel's first incompleteness theorem

Theorem

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

Consequences

- There are $2^{\aleph_0}$ different maths starting from ZF.
- This won't get any better if we change ZF for something else.
- We will never be able to construct a recursive foundation for math with first order logic where every question has an answer.
- We will never know all the truths of the arithmetic of natural numbers if we try to deduce them from an effectively generated set of axioms.

More Information

- Introduction to set Theory, Thomas Jech and Karel Hrbacek.
- Set Theory, Thomas Jech.
- Set Theory, Kenneth Kunen.