Ordinals and Cardinals
Part II

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Last Week

Definition

A set α is an ordinal if

- α is well ordered by $\in$.
- α is transitive (element implies subset).

Cantor's naive definition of ordinal numbers

- Start with the natural numbers.
- For each ordinal there is a succesor ordinal.
- Least upper bounds exist: For each set of ordinals $\{\alpha_i\}$ there is a least ordinal which is larger than them all ($\sup\{\alpha_i\}$).

Last Week

$$
\begin{aligned}
0 &:= \emptyset \\
1 &:= \{\emptyset\} = \{0\} \\
2 &:= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &:= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
4 &:= \{0, 1, 2, 3\} \\
&\vdots \\
n + 1 &:= \{0, 1, 2, \ldots, n\} = n \cup \{n\} \\
&\vdots \\
\omega &:= \bigcup n \\
\omega + 1 &:= \omega \cup \{\omega\} \\
\omega + 2 &:= \omega + 1 \cup \{\omega + 1\} \\
&\vdots \\
\omega + \omega &:= \bigcup \text{all the previous ones.}
\end{aligned}
$$

Last Week

$0 \in 1 \in 2 \in 3 \in \ldots \in \omega \in \omega + 1 \in \ldots \in \omega + \omega \in \ldots$

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- The restriction of $\in$ to any of them gives a well order.

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$$\omega\cdot 2+1,\ \omega\cdot 2+2,\ \omega\cdot 2+3,\ \ldots \omega\cdot 3,\ \omega\cdot 3+1,\ \omega\cdot 3+2,\ \ldots,\ \omega\cdot 4,$$
$$\ldots,\ \omega\cdot 5,\ \ldots,\ \omega\cdot\omega := \omega^2\ ,\ \omega^2+1,\ \omega^2+2,\ \ldots,\ \omega^2+\omega,\ \omega^2+\omega+1,$$
$$\ldots,\ \omega^2+\omega\cdot 2,\ \ldots,\ \omega^2+\omega^2 := \omega^2\cdot 2\ ,\ \omega^2\cdot 2+1,\ \omega^2\cdot 2+2,\ \ldots,$$
$$\omega^2\cdot 2+\omega,\ \ldots,\ \omega^2\cdot 3,\ \ldots,\ \omega^3,\ \omega^3+1,\ \ldots,\ \omega^4,\ \ldots,\ \omega^5,\ \ldots,\ \omega^\omega,$$
$$\omega^\omega+1,\ \ldots,\ \omega^{\omega\cdot 2}\ ,\ \omega^{\omega\cdot 2}+1,\ \ldots,\ \omega^{\omega\cdot 2}\ ,\ \ldots,\ ,\ \omega^{\omega\cdot 4},\ \ldots,\ \omega^{\omega\cdot 5}\ \ldots,$$
$$\omega^{\omega^2},\ \omega^{\omega^2}+1,\ \ldots,\ \omega^{\omega^3}\ ,\ \ldots,\ \omega^{\omega^4},\ \ldots,\ \omega^{\omega^\omega},\ \omega^{\omega^\omega}+1,\ \ldots,\ \omega^{\omega^{\omega^\omega}},$$

$$\ldots,\ \omega^{\omega^{\omega^{\omega^\omega}}}\ ,\ \ldots,\ \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}} = \varepsilon_0,\ \varepsilon_0+1,\ \varepsilon_0+2,\ \ldots,\ \varepsilon_0+\omega,\ \ldots$$

…and so on.

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…and so on. And these are all countable!

Last Week

Theorems from Set Theory

- $\in$ is a linear order on the ordinals (for any two ordinals α and β either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$).
- Successor Ordinals: If α is an ordinal, then $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal, and there are no ordinals between α and $\alpha + 1$.
- Supremum of a set of ordinals: If $\{\alpha_i\}$ is a set or ordinals then $\cup\alpha_i$ is an ordinal and is the supremum of the α_i.
- Every well ordered set is order isomorphic to a UNIQUE ordinal. (Cantor)

Last Week

$$\begin{array}{rcl}
\bullet\bullet\bullet\bullet\ldots & \cong & \omega \\
\bullet\bullet\bullet\bullet\ldots\bullet & \cong & \omega+1 \\
\bullet\bullet\bullet\bullet\ldots\bullet\bullet & \cong & \omega+2 \\
& \vdots & \\
\bullet\bullet\bullet\bullet\ldots\bullet\bullet\bullet\ldots & \cong & \omega+\omega \\
& \vdots &
\end{array}$$

Explicitly:

$1 \triangleleft 3 \triangleleft 5 \triangleleft \ldots \triangleleft 2 \triangleleft 4 \triangleleft 6 \triangleleft \ldots \cong \omega + \omega$

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Definition

A Cardinal is an ordinal which is not in bijection with any of its predecessors.

Definition

$$\begin{aligned} \aleph_0 &:= \omega \\ \aleph_1 &:= \text{The first uncountable ordinal} \end{aligned}$$

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In general for any ordinal α we define

- $\aleph_{\alpha+1} :=$ The least ordinal which is not in bijection with $\aleph_\alpha$.
- $\aleph_\alpha = \bigcup_{\delta<\alpha} \aleph_\delta$ if α is a limit ordinal (not a successor).

Last Week

Theorem

Any cardinal is one of the alephs.

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\begin{aligned}
&\aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots < \aleph_\omega := \bigcup_{i<\omega} \aleph_i < \aleph_{\omega+1} < \aleph_{\omega+2} \\
&< \ldots < \aleph_{\omega+\omega} = \aleph_{\omega\cdot 2} < \ldots < \aleph_{\omega^2} < \aleph_{\omega^2+1} < \ldots < \aleph_{\varepsilon_0} \\
&< \ldots\ldots\ldots < \aleph_{\aleph_1} < \aleph_{\aleph_1+1} < \aleph_{\aleph_1+2} < \ldots < \aleph_{\aleph_2} < \\
&\ldots < \aleph_{\aleph_3} < \ldots < \aleph_{\aleph_\omega} = \aleph_{\aleph_{\aleph_0}} < \aleph_{\aleph_{\aleph_0}+1} < \aleph_{\aleph_{\aleph_0}+2} < \ldots \\
&< \ldots\ldots\ldots < \aleph_{\aleph_{\aleph_1}} < \ldots < \aleph_{\aleph_{\aleph_2}} < \ldots\ldots < \aleph_{\aleph_{\aleph_{\aleph_0}}}
\end{aligned}
$$

$$\ldots < \aleph_{\aleph_{\aleph_{\ddots}}} \; (\aleph_0 \text{ times}) < \ldots < \aleph_{\aleph_{\aleph_{\ddots}}} \; (\aleph_1 \text{ times}) < \ldots$$

. . . and so on.

Last Week

Theorem

(Axiom of Choice implies) Every set can be well ordered, so every set has the cardinality of a unique cardinal number.

Continuum Hypothesis (Cantor)

$$| \mathbb{R} | = \aleph_1$$

or

$$2^{\aleph_0} = \aleph_1$$

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Theorem

The continuum hypothesis is independent of ZFC:

$$ZFC + CH \text{ and } ZFC + \neg CH$$

are both consistent assuming ZFC is consistent.

Last Week

Theorem (Gödel's first incompleteness theorem)

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

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- There are $2^{\aleph_0}$ different maths starting from ZF.
- This won't get any better if we change ZF for something else.
- We will never be able to construct a recursive foundation for math with first order logic where every question has an answer.

The Universe of Math

At the beginning there was nothing

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Then there was a set containing nothing

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Then there was the power set of what already existed

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The Universe of Math

The real definition

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\begin{aligned}
V_0 &:= \emptyset \\
V_{\alpha+1} &:= \wp(V_\alpha) \\
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$$\boxed{V := \text{The collection of all the } V_\alpha}$$

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Example

- $\omega \in V_{\omega+1}$.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \in V_{\omega+30}$.

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- The ordinals are the backbone of V.
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Yes. If κ is a strongly inaccessible cardinal then ZFC holds in V_κ.

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Example: $\aleph_0$ is regular and strong limit.

However....

We cannot prove the existence of a strongly inaccessible cardinal!

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- We cannot construct a strongly inaccessible cardinal using ZFC since its existence would contradict the theorem above.
- Gödel's Second Incompleteness Theorem implies ZFC cannot prove its own consistency, so believing ZFC is a consistent foundation for math is an act of faith.
- Set theorists believe that V is the place where the axioms of ZFC hold and so ZFC is consistent, but they also have a proof that they cannot prove this.

Large Cardinals

$ZFC + \exists$ Strongly Inaccessible $\Rightarrow ZFC$ is consistent

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$$ZFC + \exists \text{ Mahlo cardinal} \Rightarrow ZFC + \exists \text{ Strongly Inaccessible is consistent}$$

Large Cardinals: list (Sample)

By order of consistency strength:

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- measurable (implies something about a family of subsets of $\mathbb{R}$ being measurable).

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- $0=1$

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- There is an infintiy -larger than any infinity- number of infinities.
- Categories are huge.
- There are two types of numbers: cardinals and ordinals. For finite numbers both concepts agree, but as soon as we go to the transfinite world we can see the difference.

Cardinals: sizes

Ordinals: orders

More Information

- Introduction to set Theory, Thomas Jech and Karel Hrbacek.
- Set Theory, Thomas Jech.
- Set Theory, Kenneth Kunen.