The Genus Expansion

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The Gaussian Unitary Ensemble

The *Gaussian Unitary Ensemble* (GUE) is the space of $N \times N$ Hermitian matrices $M = (m_{ij})$ with measure

$$d\mu = \frac{1}{2^{N/2}\pi^{N^2/2}} \exp\left(-\frac{1}{2}\mathsf{Tr}M^2\right) dM$$

where dM is the Lebesgue measure on the real and imaginary parts of the matrix entries (N^2 variables)

$$dM = \prod_{i<j} d\,(\mathsf{Re}\ m_{ij})d\,(\mathsf{Im}\ m_{ij}) \prod_i dm_{ii}.$$

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The constant makes $d\mu$ a probability measure, so we can (and will) also write

$$d\mu = \frac{1}{\int_{GUE} \exp\left(-\frac{1}{2}\mathsf{Tr} M^2\right) dM} \exp\left(-\frac{1}{2}\mathsf{Tr} M^2\right) dM$$

$d\mu$ is the joint probability distribution of the N^2 independent random variables

$$\{\text{Re } m_{ij}\}_{i<j} \qquad \{\text{Im } m_{ij}\}_{i<j} \qquad \{m_{ii}\}$$

where Re m_{ij}, Im $m_{ij} \sim \mathcal{N}(0, 1/2)$ and $m_{ii} \sim \mathcal{N}(0, 1)$.

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A very important fact

For products of two matrix entries one has expectation

$$\langle m_{ij} m_{kl} \rangle = \int m_{ij} m_{kl} \, d\mu = \begin{cases} 1 & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

Wick's Lemma

If $f_1, \ldots, f_{2n}$ are $2n$ linear functions on the m_{ij}, then

$$\langle f_1 \ldots f_{2n} \rangle = \sum_{\text{couplings}} \langle f_{i_1} f_{j_1} \rangle \langle f_{i_2} f_{j_2} \rangle \ldots \langle f_{i_n} f_{j_n} \rangle$$

where a coupling of the set $\{f_1, f_2, \ldots, f_n\}$ is a partition of the set into n sets of 2 elements

$$\{f_1, f_2, \ldots, f_n\} = \{f_{i_1}, f_{j_1}\} \sqcup \{f_{i_2}, f_{j_2}\} \sqcup \ldots \sqcup \{f_{i_n}, f_{j_n}\}$$

where the ordering is not important.

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where the ordering is not important.

A way to visualize a pairing is to write the $2n$ terms next to each other

$$f_1 \quad f_2 \quad f_3 \quad \ldots f_{2n-1} \quad f_{2n}$$

and connect them in pairs by arcs.

$(\mathrm{Tr}\ M^4)^n$

In terms of the entries of the matrix we have

$$\mathsf{Tr} M^4 = \sum_{i,j,k,l=1}^{N} m_{ij} m_{jk} m_{kl} m_{li}$$

(notice the cycle present in the indices of the m's!)

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&= \left(\sum_{i_1,j_1,k_1,l_1=1}^{N} m_{i_1j_1}m_{j_1k_1}m_{k_1l_1}m_{l_1i_1}\right) \times \ldots \times \left(\sum_{i_n,j_n,k_n,l_n=}^{N}\right.
\end{aligned}$$

So...

$$\left(\mathsf{Tr}M^4\right)^n = \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n \\ k_1,\ldots,k_n \\ l_1,\ldots,l_m}=1}^{N} \begin{array}{l} (m_{i_1 j_1} m_{j_1 k_1} m_{k_1 l_1} m_{l_1 i_1})\times \\ (m_{i_2 j_2} m_{j_2 k_2} m_{k_2 l_2} m_{l_2 i_2}) \times \ldots \\ \ldots (m_{i_n j_n} m_{j_n k_n} m_{k_n l_n} m_{l_n i_n}) \end{array}$$

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which we write compactly as

$$\left(\mathsf{Tr} M^4\right)^n = \sum_{\sigma} M_\sigma$$

where $\sigma = (i_1, i_2, \ldots, i_n, j_1, \ldots, j_n, k_1, \ldots k_n, l_1, \ldots, l_n)$ runs over the N^{4n} choices for the indices from 1 to N and M_σ is defined as

$$M_\sigma = (m_{i_1 j_1} m_{j_1 k_1} m_{k_1 l_1} m_{l_1 i_1})(m_{i_2 j_2} m_{j_2 k_2} m_{k_2 l_2} m_{l_2 i_2}) \ldots (m_{i_n j_n} m_{j_n k_n} m_{l

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and to compute $\langle M_\sigma \rangle$ we may use Wick's lemma to write

$$\left\langle \left(\mathsf{Tr}M^4\right)^n \right\rangle = \sum_\sigma \sum_{\substack{\text{couplings} \\ \text{of the } 4n \\ \text{terms in } M_\sigma}} \langle \mathcal{C}_1 \rangle \ldots \langle \mathcal{C}_{2n} \rangle$$

where $\mathcal{C}_i = m_{\alpha\beta} m_{\gamma\delta}$ if $\mathcal{C}_i$ is the couple corresponding to $\{m_{\alpha\beta}, m_{\gamma\delta}\}$.

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where $\mathcal{C}_i = m_{\alpha\beta} m_{\gamma\delta}$ if $\mathcal{C}_i$ is the couple corresponding to $\{m_{\alpha\beta}, m_{\gamma\delta}\}$.

Note that this is jumbling-up the indices in a non-trivial way because of the cycles in the double indices.

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- Specifically, if $\mathcal{C} = m_{\alpha\beta} m_{\gamma\delta}$, then

$$\langle \mathcal{C} \rangle = 1 \iff \alpha = \delta \text{ and } \beta = \gamma$$

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  which is independent of the actual values of $\alpha,\beta,\gamma,\delta$ as long as the equalities hold.
- This shows that the value of $\langle\mathcal{C}_1\rangle\ldots\langle\mathcal{C}_{2n}\rangle$ only depends on equalities between the indices, and not on the specific values of the indices.

- Because of this, we may change the order of summation above to obtain

$$\left\langle \left(\mathrm{Tr} M^4\right)^n \right\rangle = \sum_{\substack{\text{couplings} \\ \text{of the } 4n \text{ double} \\ \text{indices in } M_\sigma}} \sum_{\sigma} \langle \mathcal{C}_1 \rangle \dots \langle \mathcal{C}_{2n} \rangle$$

where we are now considering the couplings of the generic $4n$ double indices

$$i_1 j_1 \quad j_1 k_1 \quad k_1 l_1 \quad l_1 i_1 \quad i_2 j_2 \quad \dots \quad i_n j_n \quad j_n k_n \quad k_n l_n \quad l_n i_n$$

and then assigning them specific values when we sum over σ.

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- In this case $\mathcal{C} = m_{\alpha\beta} m_{\gamma\delta}$ if $\mathcal{C}$ is the couple corresponding to the indices $\{\alpha\beta, \gamma\delta\}$.

Example $n = 1$

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If we compute $\left\langle \mathsf{Tr} M^4 \right\rangle$ using Wick's lemma we get

$$\left\langle \mathsf{Tr} M^4 \right\rangle = \sum_{\substack{\text{couplings} \\ \text{of the } 4 \text{ double} \\ \text{indices in } M_\sigma}} \sum_{\sigma} \langle \mathcal{C}_1 \rangle \langle \mathcal{C}_2 \rangle$$

where in this case we need to consider the couplings of the four double indices

$$ij \quad jk \quad kl \quad li$$

Example $n = 1$

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There are 3 such couplings, given by

A. $\{ij, jk\}, \{kl, li\}$

B. $\{ij, kl\}, \{jk, li\}$

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$$\left\langle \mathsf{Tr} M^4 \right\rangle = \sum_\sigma \langle m_{ij} m_{jk} \rangle \langle m_{kl} m_{li} \rangle + \sum_\sigma \langle m_{ij} m_{kl} \rangle \langle m_{jk} m_{li} \rangle + \sum_\sigma \langle m_{ij} m_{li} \rangle \langle m_{jk} m_{kl} \rangle$$

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&\quad \sum_\sigma \langle m_{ij} m_{li} \rangle \langle m_{jk} m_{kl} \rangle \\
&= \sum_\sigma \delta_{ik} + \sum_\sigma \delta_{il} \delta_{jk} \delta_{ji} \delta_{kl} + \sum_\sigma \delta_{jl} \\
&= N^3 + N + N^3 \\
&= 2N^3 + N
\end{aligned}$$

Diagrams

Fix a coupling if the $4n$ double indices in the computation of $\left\langle \left(\mathsf{Tr}M^4\right)^n \right\rangle$. Using the coupling we construct the following graph:
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- Make n vertices, corresponding to the n groups of indices (according to the parenthesis) in

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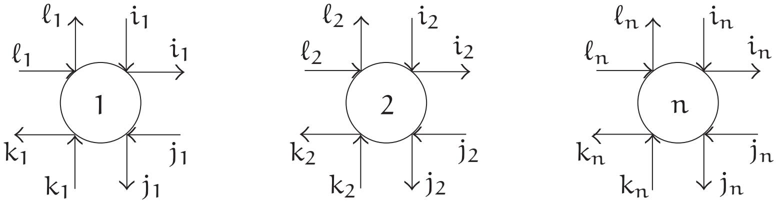
- To each vertex we assign 4 double edges, shown vertically and horizontally in the picture.



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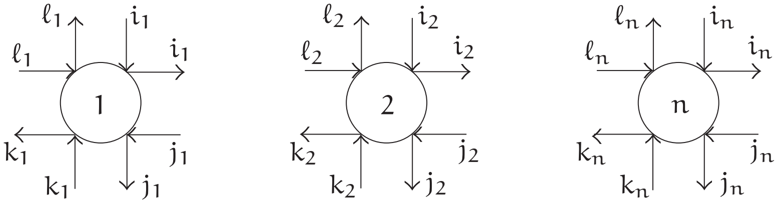

NOTE: Each double edge corresponds to an entry in the matrix.

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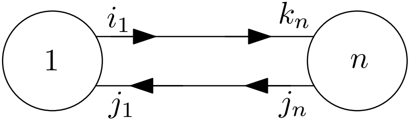

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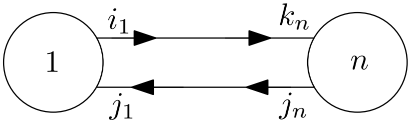

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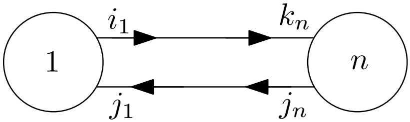
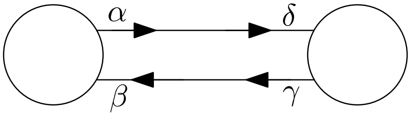


- Note: The orientations are there to encode the information that makes that pair $\langle \mathcal{C} \rangle \neq 0$: If $\mathcal{C}$ is the couple corresponding to $\alpha\beta$ and $\gamma\delta$, then $\langle m_{\alpha\beta} m_{\gamma\delta} \rangle = 1$ if and only if $\alpha = \delta$ and $\beta = \gamma$, and this will be encoded in the graph as



Diagrams

We have now constructed a labeled directed multi-graph for a given coupling of the $4n$ double indices

$$i_1 j_1 \quad j_1 k_1 \quad k_1 l_1 \quad l_1 i_1 \quad i_2 j_2 \quad \ldots \quad i_n j_n \quad j_n k_n \quad k_n l_n \quad l_n i_n$$

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From the diagram of a given coupling we can easily see which conditions on the $4n$ indices $\{i_\nu, j_\nu, k_\nu, l_\nu\}_{\nu=1}^n$ imply that

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Just follow the labels of the individual edges!

Example $n = 1$

Couplings

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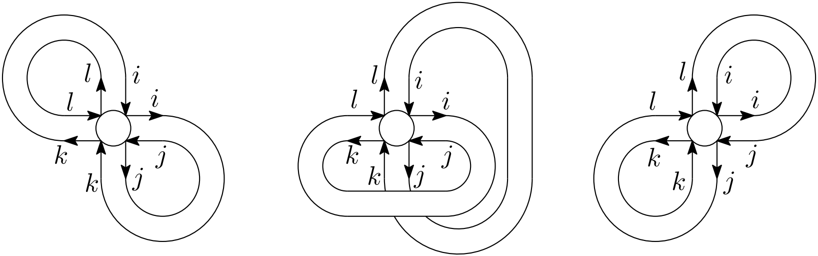
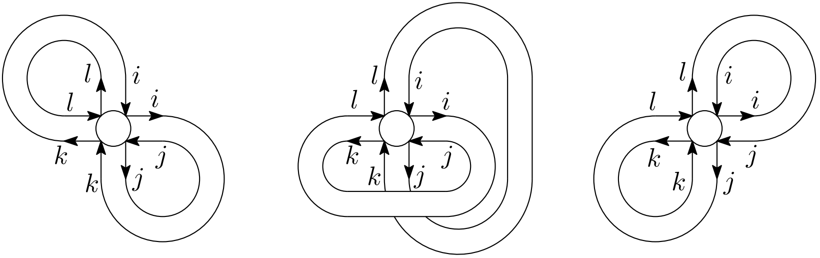
Diagrams



A. B. C.



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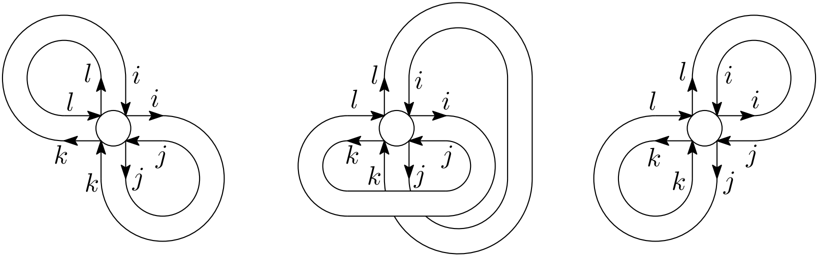
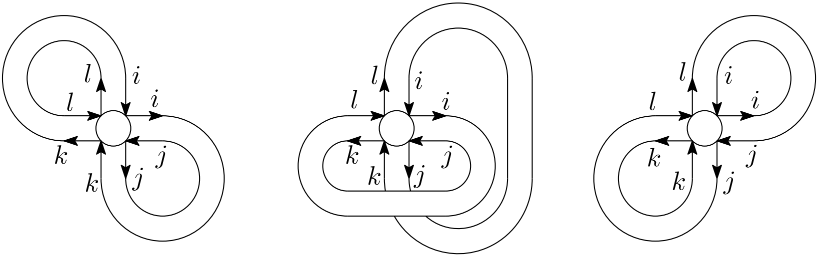


A. B. C.

For coupling A, we have the cycles

$$i \to k \to i$$
$$l \to l$$
$$j \to j$$

from which we can read the conditions $i = k$, $l = l$, $j = j$ which are the ones that make the term $\langle m_{ij} m_{jk} \rangle \langle m_{kl} m_{li} \rangle$ corresponding to the coupling be nonzero.



A. B. C.

For coupling B we have the cycle

$$i \to l \to k \to j \to i$$

from which we see that for the term $\langle m_{ij} m_{kl} \rangle \langle m_{jk} m_{li} \rangle$ corresponding to the coupling to be nonzero (and so equal to 1) we must have $i = j = k = l$.

The Faces of Diagram

In general (i.e., for arbitrary number of vertices n), for a coupling with F cycles we have

$$\sum_{\sigma} \langle \mathcal{C}_1 \rangle \ldots \langle \mathcal{C}_{2n} \rangle = N^F.$$

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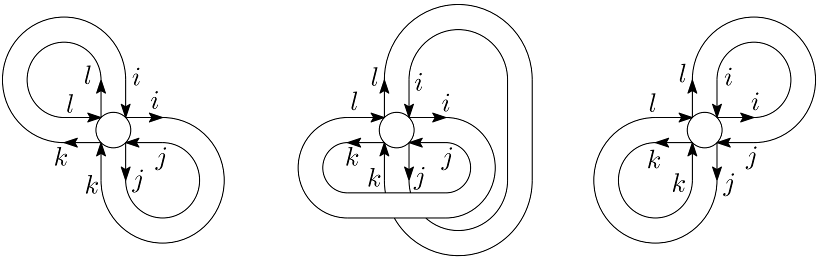
We call these cycles the faces of the diagram.

Back to the Example $n = 1$



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$$
\begin{aligned}
\left\langle \mathsf{Tr} M^4 \right\rangle &= \sum_{\sigma} \langle m_{ij} m_{jk} \rangle \langle m_{kl} m_{li} \rangle + \langle m_{ij} m_{kl} \rangle \langle m_{jk} m_{li} \rangle + \langle m_{ij} m_{li} \rangle \langle m_{j} \\
&= \sum_{\sigma} \delta_{ik} + \sum_{\sigma} \delta_{il} \delta_{jk} \delta_{ji} \delta_{kl} + \sum_{\sigma} \delta_{jl} \\
&= N^3 + N + N^3 \\
&= 2N^3 + N
\end{aligned}
$$

The general count

$$\left\langle \left(\mathsf{Tr} M^4\right)^n \right\rangle \;=\; \sum_{\substack{\text{couplings} \\ \text{of the } 4n \text{ double} \\ \text{indices in } M_\sigma}} \;\sum_{\sigma} \;\langle \mathcal{C}_1 \rangle \ldots \langle \mathcal{C}_{2n} \rangle$$

$$= \sum_{F=1}^{\infty} \left(\begin{array}{c} \text{number of} \\ \text{couplings} \\ \text{with } F \text{ faces} \end{array} \right) \cdot N^F$$

(note this sum is finite).

Diagrams and cell structures

Let Γ be a diagram from the expansion of $\left\langle \left(\mathsf{Tr} M^4\right)^n \right\rangle$ which is connected.

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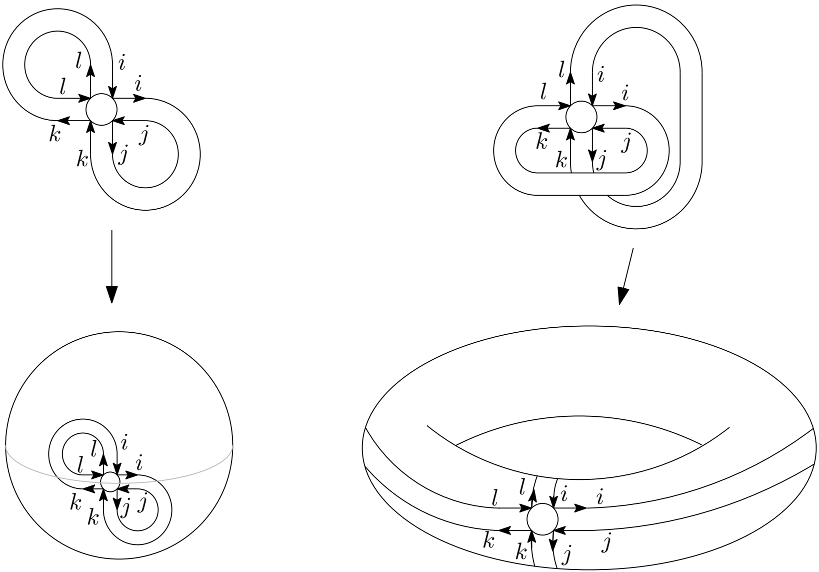
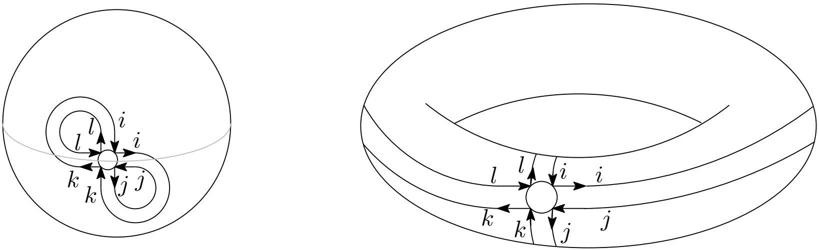
From the information in Γ we obtain a CW-complex structure for a compact orientable surface as follows:

- 0-cells: The vertices.
- 1-cells: The edges. Glued to the 0-cells as in the diagram.
- 2-cells: Discs. Glued to the 1-skeleton according to the cycles in the diagram.



This is why we called the cycles in the diagram faces!



Euler Formula

The genus of a compact orientable surface constructed from V vertices, E edges and F faces is given by

$$2 - 2g = V - E + F$$

So, we can view Γ as a (multi-)graph which is embedded inside a compact orientable surface.

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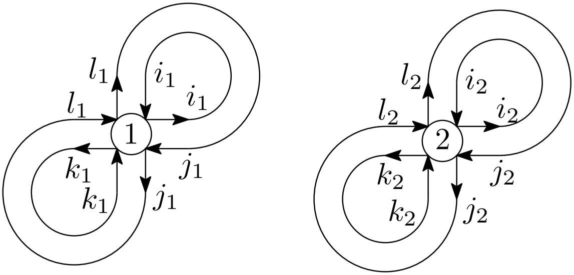
- If we use the Euler formula to compute the "genus", then this genus may now be negative!
- e.g. A coupling from $\left\langle \left(\mathrm{Tr} M^4\right)^2 \right\rangle$ with genus -1:



The Bijection

We have seen the bijection

$$\left\{ \begin{array}{c} \text{couplings} \\ \text{from } \left\langle \left(\text{Tr} M^4\right)^n \right\rangle \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{labeled diagrams} \\ \text{with } n \\ 4\text{-valent vetices} \end{array} \right\}$$

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In the last term we can decide to forget some information and still maintain the bijection.

Example

There is a bijection between the couplings from $\left\langle \left(\mathrm{Tr} M^4\right)^n \right\rangle$ and embedded 4-valent graphs with n vertices where:

- The complement of the graph is a disjoint union of sets homeomorphic to discs.
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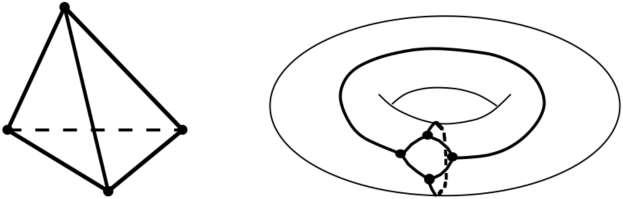
- The special edge tells us which is the double edge with labels i_1, j_1 corresponding to the matrix entry m_{i_1,j_1}.

Why the orientations are important

Without the labels (which induce orientations), from an abstract (multi-)graph we can get cell structures for very different surfaces!



(Image from Zvonkin's "Matrix Integrals and Map Enumeration")

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This number is NOT easy to compute!

Back to the count

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&= \sum_{g=-\infty}^{\infty} \begin{pmatrix} \text{number of labeled} \\ \text{graphs with } n \\ 4\text{-valent vertices embedded} \\ \text{in a (collection of)} \\ \text{genus } g \text{ surface(s)} \end{pmatrix} \cdot N^{2-2g+n}
\end{aligned}$$

The General Case (arbitrary valence)

Expectations of the form

$$\left\langle \left(\mathsf{Tr} M^1\right)^{n_1} \left(\mathsf{Tr} M^2\right)^{n_2} \dots \left(\mathsf{Tr} M^\nu\right)^{n_\nu} \right\rangle .$$

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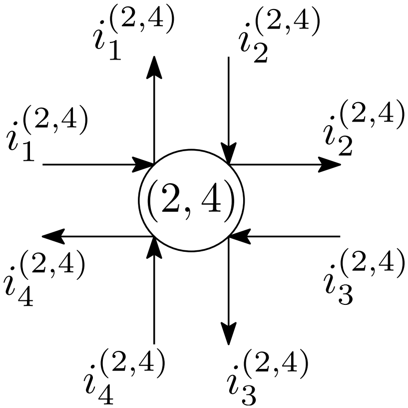
will correspond (via Wick's lemma) to diagrams with n_1 vertices of valence 1, n_2 vertices of valence 2 and so on.
The vertices are numbered by tuples

$$(a, b) = (\text{vertex } \#, \text{valence}),$$

where $b = 1, \ldots, \nu$ and $a = 1, \ldots, n_b$ (it is understood that if $n_j = 0$ then there are no vertices of valence j), and around each vertex we place the corresponding number of edges, and label the vertex (a, b) clockwise by

$$i_1^{(a,b)}, i_2^{(a,b)}, \ldots, i_b^{(a,b)}.$$



Example: The second vertex of valence 4

By following the same arguments as above for the case of 4-valent diagrams one may show that

$$\left\langle \left(\mathrm{Tr}M^1\right)^{n_1} \left(\mathrm{Tr}M^2\right)^{n_2} \ldots \left(\mathrm{Tr}M^\nu\right)^{n_\nu} \right\rangle = \sum_{F=1}^{\infty} \left(\begin{array}{c} \text{number of} \\ \text{couplings} \\ \text{with } F \text{ faces} \end{array} \right) \cdot N^F$$

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The strange exponent in the N

- There are $V = \sum n_j$ vertices, and $E = \frac{1}{2}\sum j n_j$ edges.
- $2 - 2g = V - E + F$ gives F (the number of cycles in the coupling or the diagram) in terms of the other quantities.

The Generating Function

If we set

$$F(t_1, \ldots, t_\nu) = \left\langle \exp\left(t_1 \mathsf{Tr} M^1 + t_2 \mathsf{Tr} M^2 + \ldots + t_\nu \mathsf{Tr} M^\nu\right)\right\rangle,$$

then we can recover all the expectations by noting that

$$\left.\frac{\partial^n}{\partial t_1^{n_1} \ldots \partial t_\nu^{n_\nu}} F\right|_{t=0} = \left\langle \left(\mathsf{Tr} M^1\right)^{n_1} \left(\mathsf{Tr} M^2\right)^{n_2} \ldots (\mathsf{Tr} M^\nu)^{n_\nu} \right\rangle,$$

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where we are interpreting the derivative of F as the formal derivative under the integral sign.

Note: We are not claiming that F is differentiable at $t = 0$ (in fact, sometimes F is undefined if $t \neq 0$), and this should just be interpreted as a formal "packaging" of all the quantities $\left\langle \left(\mathsf{Tr} M^1\right)^{n_1} \left(\mathsf{Tr} M^2\right)^{n_2} \ldots \left(\mathsf{Tr} M^\nu\right)^{n_\nu}\right\rangle$.

Another way to interpret $F(t_1, \ldots, t_\nu)$ is to view it as the formal generating function of the $\left\langle \left(\mathrm{Tr} M^1\right)^{n_1} \left(\mathrm{Tr} M^2\right)^{n_2} \ldots \left(\mathrm{Tr} M^\nu\right)^{n_\nu} \right\rangle$ by using the expansion of the exponential

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$$\begin{aligned}
\exp(a_1 t_1 + \ldots + a_\nu t_\nu) &= \sum_{n \geq 0} \frac{(a_1 t_1 + \ldots + a_\nu t_\nu)^n}{n!} \\
&= \sum_{n \geq 0} \frac{1}{n!} \sum_{n_1 + \ldots + n_\nu = n} \frac{n!}{n_1! \ldots n_\nu!} \prod (a_j t_j)^{n_j} \\
&= \sum_{n \geq 0} \sum_{n_1 + \ldots + n_\nu = n} \frac{\prod_{j=1}^{\nu} a_j^{n_j}}{n_1! \ldots n_\nu!} t_1^{n_1} \ldots t_\nu^{n_\nu} \\
&= \sum_{n_1, \ldots, n_\nu \geq 0} \frac{\prod_{j=1}^{\nu} a_j^{n_j}}{n_1! \ldots n_\nu!} t_1^{n_1} \ldots t_\nu^{n_\nu},
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If we use this in the the integral in

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and then formally commute integrals and sums we obtain

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and again, does not mean F is differentiable!

The Partition Function

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- We clean up the exponents on the right by taking them to the left

$$\left\langle \prod_{j=1}^{\nu} \frac{1}{N^{\frac{1}{2}(j-2)n_j}} \left(\mathsf{Tr} M^j\right)^{n_j} \right\rangle = \sum_{g\in\mathbb{Z}} A_{g,n_1,\ldots n_\nu} N^{2-2g}$$

- And let $Z_N(-t)$ be the generating function of these Laurent polynomials in N.

The Partition Function

$$\widehat{Z}_N(t) = \left\langle \exp\left(-\sum_{j=1}^{\nu} N^{-\frac{1}{2}(j-2)} t_j \mathsf{Tr} M^j \right) \right\rangle,$$

giving the formal expansion

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Remember:

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The Partition Function

$$\widehat{Z}_N(t) = \frac{\int \exp\left(-\sum_{j=1}^{\nu} N^{-\frac{1}{2}(j-2)} t_j \mathsf{Tr} M^j\right) e^{-\frac{1}{2}\mathsf{Tr} M^2} dM}{\int e^{-\frac{1}{2}\mathsf{Tr} M^2} dM}$$

Make the substitution $M = \sqrt{N}\widehat{M}$ in both integrals to obtain (with $\widehat{M}$ instead of M, but I drop the hat below)

$$\widehat{Z}_N(t) = \frac{\int \exp(-N \mathsf{Tr}(V_t(M)))dM}{\int \exp(-N \mathsf{Tr}(V_0(M)))dM}$$

where

$$V_t(M) = \frac{1}{2}M^2 + \sum_{j=1}^{\nu} t_j M^j.$$

The magic of the log

The magic of the log

$$\log \widehat{Z}_N(t) \text{``} = \text{''} \sum_{n_1,\ldots,n_\nu \geq 0} (-1)^{\sum n_j} \frac{P_{n_1,\ldots,n_v}(N)}{n_1! \ldots n_\nu!} t_1^{n_1} \ldots t_\nu^{n_\nu}$$

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... Now reorder the sum on the right ...

The Genus Expansion

$$\frac{1}{N^2}\log \widehat{Z}_N(t) \text{``}=\text{''} \sum_{g\geq 0} e_g(t)\frac{1}{N^{2g}},$$

where

$$e_g(t) = \sum_{n_1,\dots,n_\nu \geq 0} (-1)^{\sum n_j} \frac{\kappa_g(n_1,\dots,n_\nu)}{n_1!\dots n_\nu!} t_1^{n_1}\dots t_\nu^{n_\nu}$$

and $\kappa_g(n_1,\dots,n_\nu)$ is the number of of connected labeled diagrams of genus g with n_j-vertices of valence j.

A Precise Mathematical Interpretation

$$\frac{1}{N^2} \log \widehat{Z}_N(t) \text{``}=\text{''} \sum_{g \geq 0} e_g(t) \frac{1}{N^{2g}},$$

Ercolani and McLaughlin, 2003:

It ν is even, then there is a in cone $\Omega \subseteq \mathbb{R}^\nu$ with vertex at the origin for the t's for which $\log \widehat{Z}_N(t)$ is a differentiable function of t, and there is an $N_0 > 0$ such that for all $G > 0$ there exists a constant C_G such that

$$\left| \frac{1}{N^2} \log \widehat{Z}_N(t) - \left(e_0(t) + \frac{e_1(t)}{N^2} + \ldots + \frac{e_G(t)}{N^{2G}} \right) \right| < \frac{C_G}{N^{2G+2}}$$

for all $t \in \Omega$ and $N > N_0$, and the same holds for the partial derivatives of $\log \widehat{Z}_N(t)$ (with possibly different constants C_G).

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for all $t \in \Omega$ and $N > N_0$, and the same holds for the partial derivatives of $\log \widehat{Z}_N(t)$ (with possibly different constants C_G). Note, this does not imply equality since the constant C_G depends on G. It does imply nonetheless that both quantities get closer as $N \to \infty$ uniformly for $t \in \Omega$ and fixed G.