Lines on cubic surfaces, elliptic surfaces and the $E_6$ lattice

T. Shioda, *Weierstrass Transformations and Cubic Surfaces*

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Theorem

Any smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.
Cubic Surfaces

Theorem

Any smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

However...the lines are usually not defined over the reals.

NOTE: These are not smooth...but you get the point.
Cubic Surfaces

The Clebsch Diagonal Cubic

\[ x^3 + y^3 + z^3 + w^3 - (x + y + z + w)^3 = 0 \]
Blowing up

- Given any smooth algebraic surface, replace a point by a line (the Exceptional Line): a copy of $\mathbb{P}^1$.
- Each point on this line corresponds a tangent direction at the point.
Cubic Surfaces and blow up’s

- Every smooth cubic in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$ blown up at six points (not all on a conic and no three on a line).
- The 27 lines are given by:
Cubic Surfaces and blow up’s

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  15: The strict transform of the lines connecting any two of the six points.
  6: The strict transform of the 6 (unique) conics through five of the six points.
Two natural questions

**Question 1**

Given a cubic smooth surface, can you find the equations of the 27 lines?
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Given a cubic smooth surface, ¿Can you find the equations of the 27 lines?

**Question 2**
Given six points in $\mathbb{P}^2$, ¿Can find the equation of the corresponding smooth cubic surface and the 27 lines in it?
Very difficult!...the lines are usually not defined over the field of definition of the surface.
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Very difficult!...the lines are usually not defined over the field of definition of the surface.

Example (Shioda)
The minimum field extension of $\mathbb{Q}$ where all the 27 lines of the cubic surface

$$y^2 + 2yz = x^3 + x + xz^2 + z + z^2 + 1$$

are defined is the splitting field of a polynomial of degree 27. The degree of this extension is 51,840.
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More Generally (Harris)
There are no explicit equations for the 27 lines of a general cubic surface.
Question 2:
Given the six points you blow up, can you find the equation of the resulting smooth cubic surface and the 27 lines in it?
The space of plane cubics is $9$-dimensional ($10$ projective coefficients).

Assuming the six points are not on a conic and no three are on a line, the space of cubics through the six points is $3$-dimensional ($9 - 6 = 3$), i.e., the conditions are independent.

Let $f_0, f_1, f_2, f_3$ be a basis. The equation of any plane cubic through the six points is a linear combination of the $f_i$.

Define the rational map $\mathbb{P}^2 \to \mathbb{P}^3 [x:y:z] \mapsto [f_0(x, y, z): \ldots : f_3(x, y, z)]$.

The map is not defined at the six points since all the $f_i$ give zero, but it extends to a map from the blowup of $\mathbb{P}^2$ at those points to $\mathbb{P}^3$ which is in fact an ISOMORPHISM.
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The Article

- Explicit equation for the cubic in terms of the equations of the blown-up points.
- The construction involves the $E_6$ lattice and its dual $E_6^*$
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The answer

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The cuspidal cubic has a parametrization $(1/u^2, 1/u^3)$, so each $P_i$ corresponds to a $u_i$.

The condition on their configuration is equivalent to:

- $u_i \neq u_j$ if $i \neq j$.
- $\sum u_i \neq 0$
- $u_i + u_j + u_k \neq 0$ for $i, j, k$ distinct.
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Let $c_n$ be the $n$-th symmetric function in the $u_i$:
$$\prod (x - u_i) = x^6 - c_1x^5 + c_2x^4 + \ldots + c_6$$
The answer

Let $\varepsilon_n$ is the $n$-th symmetric function in the 27 forms:

\[
a_i = \frac{c_1}{3} - u_i \quad a'_i = \frac{-2c_1}{3} - u_i \quad a''_{ij} = \frac{c_1}{3} - u_i - u_j
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The equation of the cubic surface obtained by blowing up \( P_1, \ldots, P_6 \) is

\[
y^2 + 2y = x^3 + x(p_0 + p_1z + p_2z^2) + (q_0 + q_1z + q_2z^2)
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where

$$p_2 = \frac{\varepsilon_2}{12}$$
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$$p_1 = \varepsilon_5/48$$
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q_2 &= (\varepsilon_6 - 168p_2^3)/96 \\
p_0 &= (\varepsilon_8 - 294p_1p_2^2)/1344 \\
q_0 &= (\varepsilon_{12} - 608p_1^2p_2 - \ldots + 1248q_2^2)/17280
\end{align*}
$$
The answer

The equations for the 27 lines are also explicit:

\[ x = az + b \quad \cap \quad y = dz + e \]

\[ L_i: \ a = a_i. \]
\[ L'_i: \ a = a'_i. \]
\[ L'_{ij}: \ a = a'_{ij}. \]

\[ b = \text{complicated expression in } c_1, c_2, c_3, c_4, u_i \]

In all cases:

\[ d = (a^3 + ap_2)/2 \]
\[ e = (3a^2b - d^2 + ap_1 + bp_2 + q_2)/2 \]
Theorem

The Mordell-Weil lattice of the elliptic curve $E$: $y^2 = x^3 + x(p_0 + p_1 t + p_2 t^2) + (q_0 + q_1 t + q_2 t^2 + t^4)$ defined over $\mathbb{Q}(t)$ is isomorphic to $E^\ast_6$.

Moreover,

\begin{align*}
\text{The 54 minimal roots are given by 27 points } P = (x, y) \\
\text{of the form } x = a + by, y = t^2 + dt + e.
\end{align*}

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\text{The coefficients } a, b, c, d \text{ can be given explicitly in terms of the } p_i, q_j.
\end{align*}

These roots generate the Mordell-Weil group, and one can give six explicit points which generate the Mordell-Weil group.
¿How?

**Theorem**

The Mordell-Weil lattice of the elliptic curve

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- The 54 minimal roots are given by 27 points \( P = (x, y) \) of the form

  \[ x = at + b \quad y = t^2 + dt + e \]

  and the 27 negatives of the points above \( -P = (x, -y) \).
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- The map \( y \mapsto y - t^2 \) maps this surface to a cubic surface and the root vectors of the form \( P = (at + b, t^2 + dt + e) \) to lines \( (at + b, dt + e) \).
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- So we have a cubic surface and the 27 lines on it.
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- Now relate this somehow to the blowup of six points in the plane....
The 28 bitangents
The grey cubics:
By Oliver Labs, from his webapge The Cubic Surface Homepage at http://www.cubics.algebraicsurface.net/

The Clebsh Cubic (blue):
By Stephan Endrass, made with his graphing program SURF.

The 28 bitangents:
Apparently hand drawn by T. Shioda, from his paper Weierstrass Transformations and Cubic Surfaces.